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Spectral distributions for long range perturbations

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Abstract

We study distributions which generalize the concept of spectral shift function, for pseudo-differential operators on $\mathbb{R}^d$. We call such distributions *spectral distributions*. Relations between relative scattering determinants and spectral distributions are established; they lead to the definition of regularized scattering phase. These relations are analogous to the usual one for the standard spectral shift function. We give several asymptotic properties in the high energy and semiclassical limits where both nontrapping and trapping cases are considered. In particular, we prove Breit–Wigner formulae for the regularized scattering phases, for semiclassical Schrödinger operators with long-range potentials.

0. Introduction

0.1. Statement of the problem

The spectral shift function (SSF in the sequel) of the pair of self-adjoint operators (H_0, H_1) is defined, under some conditions on $V = H_1 - H_0$ which we discuss below, as the locally integrable function s_1 such that

$$\operatorname{Tr}(f(H_1) - f(H_0)) = -\int_{\mathbb{R}} f'(\lambda) s_1(\lambda)\, d\lambda, \quad f \in C_0^\infty(\mathbb{R}). \tag{0.1}$$

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If the operators are semibounded from below, which we shall always assume, s_1 is uniquely defined by the condition $s_1(\lambda) = 0$ for $\lambda \ll 0$. The SSF was first studied by Krein [27] from the mathematical point of view. It is a fundamental object in scattering theory with several physical meanings (time delay, scattering phase) and various mathematical applications (including geometry, see [15,21]). For a rather complete survey on these applications, we refer to [3,35] (see also [18] and the book [38]).

The SSF can be considered as a *scattering phase* for the following two reasons:

$$\mathrm{Det}_1\, S(\lambda) = e^{2i\pi s_1(\lambda)}, \quad \text{a.e. on } \sigma_{ac}(H_0), \tag{0.2}$$

$$s_1(\lambda) = -\frac{1}{\pi} \lim_{\varepsilon \searrow +0} \arg \mathrm{Det}_1(1 + VR_0(\lambda + i\varepsilon)), \quad \text{a.e. on } \mathbb{R}. \tag{0.3}$$

The first one, which justifies historically the terminology, is known as Krein's formula and was proved by Birman–Krein [2]. Here $S(\lambda)$ is the scattering matrix of (H_0, H_1) defined almost everywhere on $\sigma_{ac}(H_0)$, the absolutely continuous spectrum of H_0 (see [38]). In the second formula $R_0(z) = (H_0 - z)^{-1}$ is the free resolvent and Det_1 is the Fredholm determinant defined for *trace class perturbations* of identity (see Section 1 for the determinants). These results hold if $V \in \mathbf{S_1}$, the Schatten ideal of order 1 of trace class operators. Actually (0.1) and (0.2) are still valid provided $(H_1 + E)^{-N} - (H_0 + E)^{-N} \in \mathbf{S_1}$ for some E and N large enough, since one can use the Birman–Kato invariance principle. This principle allows to replace H_j by $\tilde{H}_j = (H_j + E)^{-N}$ in the sense that $s_1(\lambda) = -\tilde{s}_1((\lambda + E)^{-N})$ is the SSF of (H_0, H_1) if $\tilde{s}_1$ is the SSF of $(\tilde{H}_0, \tilde{H}_1)$.

If $H_0 = -\Delta$ and $H_1 = -\Delta + V$ are Schrödinger operators on $\mathbb{R}^d$, the SSF can only be defined for V with sufficiently fast decay at infinity. A natural condition which ensures the existence of the SSF, is that for all α

$$\partial^\alpha V(x) = \mathcal{O}(\langle x \rangle^{-\rho}), \qquad \rho > d, \tag{0.4}$$

with the usual notation $\langle x \rangle = (1 + |x|^2)^{1/2}$. Weaker conditions on the smoothness of V can be considered (see for example [32]), but the integrability of V at infinity is always needed.

The goal of this article is to relax the decay assumption on the perturbation, and thus to be able to consider *long-range* potentials, i.e. potentials satisfying (0.4) for some $\rho > 0$. In that case, one can always find an integer p such that $p\rho > d$ and for such a p, it is easy to prove that, for some N and E

$$(H_1 + E)^{-N} - (H_0 + E)^{-N} \in \mathbf{S_p}, \tag{0.5}$$

where $\mathbf{S_p}$ is the Schatten class of order p (see Section 1). Thus it is natural to deal with $\mathbf{S_p}$-perturbations or *relatively*-$\mathbf{S_p}$-*perturbations* (i.e. such that (0.5) holds). Under such more general conditions, the SSF s_1 cannot be defined since $\mathbf{S_1} \subsetneq \mathbf{S_p}$ if $p>1$, and some regularization has to be done.

The regularization that we shall use is based on Taylor's formula. If H_0 is a general self-adjoint operator, and $V \in \mathbf{S_p}$ (self-adjoint too), Koplienko [25] has proposed a way of regularizing formula (0.1) by considering a functional u_p defined as follows:

$$\langle u_p, f \rangle = \mathrm{Tr}\left(f(H_0+V) - \sum_{j=0}^{p-1} \frac{1}{j!}\frac{d^j}{d\varepsilon^j} f(H_0+\varepsilon V)_{|_{\varepsilon=0}} \right). \tag{0.6}$$

In Koplienko's work, the functions f are rational fractions vanishing at infinity without poles on the spectra of the operators. It is clear that the operator considered on the right-hand side of (0.6) is trace class, since the ideal $\mathbf{S_p^p} = \mathbf{S_1}$. Indeed, if $f_z(\lambda) = (\lambda - z)^{-1}$, we get

$$\langle u_p, f_z \rangle = (-1)^p \,\mathrm{Tr}((H_0+V-z)^{-1}(V(H_0-z)^{-1})^p). \tag{0.7}$$

However, as mentioned in [25], one cannot use the invariance principle any longer in (0.6) for $p \geqslant 2$: this means that, if we only assume that (0.5) holds, we cannot define a distribution u_p by replacing the operators by some power of their resolvents. In particular, Koplienko's result cannot be applied to differential operators (except for the special case $p=2$, see [25,26], see also [35]). This problem has been considered by the author in [4,7], where the following result is proved:

Theorem 0.1. *Let $H_0 = p(x,D)$ and $H_0 + V = H_0 + v(x,D)$ be self-adjoint differential operators on $\mathbb{R}^d$ with smooth bounded coefficients (as well as their derivatives) and uniformly elliptic (i.e. their principal symbols are lower bounded on $\mathbb{R}^d \times S^{d-1}$ by some $c>0$). Moreover, we suppose*

$$|\partial^\beta v_\alpha(x)| \leqslant C\langle x\rangle^{-\rho-\kappa|\alpha|}, \quad \rho > d/p, \tag{0.8}$$

with $\kappa \in [0,1]$, and $v(x,D) = \sum_{|\alpha|\leqslant 2m} v_\alpha(x) D^\alpha$.

Then for all $f \in \mathscr{S}(\mathbb{R})$, formula (0.6) makes sense and it defines a temperate distribution u_p, called spectral distribution of order p.

Remark that $u_p \equiv 0$ below the spectra of the operators, which is obvious by formula (0.6).

We emphasize that H_0 might have variable coefficients and that V might be of the same order as H_0. This theorem is actually proved for a wider class of pseudo-differential operators in [4,7]; we quote it in this less technical setting, which is sufficient for stating our main results in the next subsection.

Several natural questions arise about the distributions u_p associated with pseudo-differential operators. If one looks at formula (0.3), it is natural to study relations between u_p and some scattering determinants. Moreover, for general $\rho>0$, it is known that the scattering matrix $S(\lambda)$ may not be a compact perturbation of identity, and its spectrum can be the whole unit circle (see [39]). Thus, one cannot hope to compute any determinant of $S(\lambda)$ in the general long-range situation, and we may ask what should play the role of a *scattering phase* in this case. In order to address these questions, it seems rather natural to consider the *relative scattering determinants* defined for $\mathbf{S_p}$-perturbations of identity (see Section 1):

$$D_p(z) = \mathrm{Det}_p(1 + VR_0(z)). \tag{0.9}$$

Determinants like (0.9) are of special interest since they do not induce any distinction between the short-range, $\rho > 1$, and long-range conditions, $\rho > 0$. However, recall that the perturbations V that we consider are in general H_0-bounded and not H_0-compact, thus the meaning of $D_p(z)$ is not clear. The definition of (0.9) uses the meromorphic continuation of a so-called *regularized zeta function* as it can be done for the SSF. This is a rather usual process and we refer to the nice paper of Müller [29] on this subject (see also [8,12]).

Finally we mention that another kind of regularization based on Taylor's formula too, has been considered by Melin in [28]. Recent results associated to the related regularized scattering phase, in connection with some heat invariants, have been announced by Hitrik–Polterovich [23].

0.2. Main results of this article

We are going to prove results partially announced in [6]. In all what follows, $\rho > 0$ is a fixed positive real number and $p \geqslant 1$ an integer such that

$$\rho > d/p.$$

In the following theorem, we define the determinants $D_p(z)$ and the associated scattering phase s_p for H_0-bounded perturbations V. We consider the same differential operators H_0 and $H_0 + V$ as in Theorem 0.1 with $\kappa = 0$. We shall use the so-called *regularized Zeta function* defined by

$$\zeta_s(z) = \langle u_p, (. - z)^{-s} \rangle$$

where $(. - z)^{-s}$ is the function $\lambda \mapsto (\lambda - z)^{-s}$.

Theorem 0.2. (i) *$\zeta_s(z)$ is well defined for* $\mathrm{Re}\, s \gg 1$ *and* $z \in \mathbb{C}\backslash[\inf \mathrm{supp}\, u_p, \infty)$. *It has a meromorphic continuation, with respect to s, to the complex plane. This continuation is regular at* $s = 0$.

(ii) *The function* $D_p(z) := \exp(-\partial_s \zeta_s(z)|_{s=0})$ *is holomorphic on* $\mathbb{C}\backslash[\inf \mathrm{supp}\ u_p, \infty)$ *and the following limit holds*:

$$\lim_{\delta \searrow 0} \frac{d}{d\lambda} \arg D_p(\lambda + i\delta) = -\pi u_p(\lambda), \quad \text{in } \mathscr{D}'(\mathbb{R}), \tag{0.10}$$

where u_p is defined by (0.6).

Notice that H_0 can have variable coefficients.

Remark. Our definition of $D_p(z)$ coincides with the usual one if $V(H_0 - z)^{-1} \in \mathbf{S_p}$ (see Section 1).

Note moreover that the convergence in (0.10) is not pointwise in general, since it was proved in [4,7] that $u_2 = \eta''$ (in the distributions sense) for some locally integrable function η, called Koplienko's function (see also [25]). Formula (0.10) can be viewed as a generalization of (0.3), and it motivates the following definition:

Definition 0.3. The regularized scattering phase $s_p \in \mathscr{S}'(\mathbb{R})$ is the unique primitive of u_p which vanishes near $-\infty$.

This definition makes sense since u_p vanishes below the spectra of the operators,. Moreover, if $p = 1$, we get the usual scattering phase i.e. Birman–Krein's SSF.

Now we state results in Euclidean scattering, in the high energy and semiclassical limits.

Theorem 0.4. *Assume that* $H_0 = p_0(D)$, *with* $p_0(\xi) = \sum_{|\alpha|=2m} p_\alpha \xi^\alpha$ *constant coefficients elliptic positive polynomial*, *and consider* V *as in Theorem* 0.1 *with* $\kappa = 1$. *Then*

(i) s_p *is* C^∞ *on* $(0, \infty)\backslash\sigma_{pp}(H_0 + V)$, *where* $\sigma_{pp}(H_0 + V)$ *is the point spectrum of* $H_0 + V$.

(ii) *If the classical flow is nontrapping*, *then the following complete high energy expansion holds*:

$$s_p'(\lambda) \sim \lambda^{\frac{d}{2m}-1} \sum_{k \geqslant 0} a_k \lambda^{-k/m}, \quad \lambda \nearrow +\infty.$$

Moreover this expansion can be differentiated at any order.

Recall that the classical flow is nontrapping if $|\Pi(\phi^t(x,\xi))| \to \infty$, as $|t| \to \infty$, uniformly on each compact subset of $T^\star\mathbb{R}^d\backslash 0$, where ϕ^t is the Hamiltonian flow of the principal symbol of $H_0 + V$, and $\Pi : T^\star\mathbb{R}^d \to \mathbb{R}^d$ is the projection. Here the principal symbol is the classical one, that is the leading part of degree $2m$ of the full symbol of $H_0 + V$.

This theorem is well known for $p = 1$ [11,13,20,30,33,34] and was proved by the author for $p = 2$ [4,5,7], but the result is completely new for $p \geqslant 3$. A similar result has been announced by Hitrik–Polterovich in [23] with Melin's regularization [28] for a potential perturbation for which the nontrapping condition is always fulfilled.

Now we quote our results for the semiclassical Schrödinger operator with a long-range potential. Assume that $H_0 = -h^2\Delta$ and that $V = V(x)$ is a potential such that

$$\partial^\alpha V(x) = \mathcal{O}(\langle x \rangle^{-\rho-|\alpha|}), \quad \alpha \in \mathbb{N}^d, \tag{0.11}$$

and let $I \subset (0, \infty)$ be an interval. This interval is said to be nontrapping if $|\Pi(\phi^t(x,\xi))| \to \infty$, as $|t| \to \infty$, uniformly on each compact subset of $p^{-1}(I)$, where $p(x,\xi) = \xi^2 + V(x)$ is the semiclassical principal symbol of $H_0 + V$.

Theorem 0.5. *Assume that I is noncritical for V.*

(i) *If I is nontrapping, we have the following complete asymptotic expansion in $C^\infty(I)$:*

$$s_p'(\lambda, h) \sim h^{-d} \sum_{k \geqslant 0} h^k a_{k,p}^V(\lambda), \quad h \searrow 0, \;\; a_{k,p}^V \in C^\infty(I), \;\; k \in \mathbb{N},$$

where, if we set $c_d = (2\pi)^{-d}\mathrm{vol}(\mathbb{S}^{d-1})/2$, the leading coefficient is

$$a_{0,p}^V(\lambda) = c_d \int (\lambda - V(x))_+^{\frac{d-2}{2}} - \sum_{j=0}^{p-1} (-1)^j (d/2-1)\cdots \times (d/2-j)\lambda^{\frac{d}{2}-1-j} V(x)^j / j! \, dx, \tag{0.12}$$

with the convention that $(d/2-1)\cdots(d/2-j) = 1$ if $j = 0$. Moreover $(t)_+ = \max(t, 0)$ and if $d = 2$ then $(t)_+^{d/2-1} = 1$ if $t > 0$, and 0 otherwise.

(ii) *Assume that the following estimates holds for some $n \geqslant 1$ and $s > 0$, locally uniformly on I,*

$$\langle x \rangle^{-s}(H_0 + V - \lambda \pm i0)^{-1}\langle x \rangle^{-s} = \mathcal{O}(\exp(h^{-n})), \quad \lambda \in I. \tag{0.13}$$

Then, for all $v \geqslant 0$, the Riesz mean of order v has the following expansion, locally uniform on I

$$\int_{-\infty}^{\lambda} s_p'(\mu, h)(\lambda - \mu)^v \, d\mu = h^{-d} \sum_{k=0}^{[v]_+} C_{k,V}^v(\lambda) h^k + \mathcal{O}(h^{1+v-d}), \tag{0.14}$$

where $[v]_+$ is the smallest integer $\geqslant v$. In particular, if $v = 0$, we get the Weyl formula

$$s_p(\lambda, h) = h^{-d} C_{0,V}^0(\lambda) + \mathcal{O}(h^{1-d}).$$

Note that the Riesz mean, defined as the left-hand side of (0.14), must be understood as

$$\langle f u_p, (\lambda - .)^v \rangle + \int_{-\infty}^{\lambda} (1-f)(\mu) u_p(\mu, h)(\lambda - \mu)^v \, d\mu,$$

where $C_0^\infty(\mathbb{R}) \ni f \equiv 1$ near $[\inf \sigma(H_0 + V), 0]$. The above integral makes sense because $s_p'(., h) = u_p(., h)$ is smooth on $(0, \infty)$ since $\sigma_{pp}(H_0 + V) \cap (0, \infty)$ is empty by Arai and Uchiyama [1] and Robert [34].

We emphasize that if d is even, all the terms corresponding to $j \geqslant d/2$ vanish in (0.12) (if ever $d/2 \leqslant p-1$). In particular, if $d=2$, the leading term is the same as for the usual SSF, and more generally $a^V_{0,p} = a^V_{0,d/2}$ for all $p \geqslant d/2$.

Finally let us remark that sufficient conditions leading to the estimates (0.13) can be found in the work of Burq [10], combined with the one of Bruneau–Petkov [9].

In the next theorem, we prove that the phase shift $\pm 1/2$ which occurs for the SSF s_1 close to a resonance, holds for all s_p. This phenomenon is known as Breit–Wigner formula (see the book [37] for its physical interpretation) and has already been considered by several authors. Let us quote the papers of Combes–Duclos–Klein–Seiler [14], Petkov–Zworski [31], and especially the ones of Gérard–Martinez–Robert [17] and Robert [34], which we have followed.

We consider an energy level $\lambda_0 \in I$ (which can be critical for V) and the set of resonances $\Gamma(h)$ close to λ_0 defined by Helffer–Sjöstrand in [22]. We use suppose, as in [17], that:

- there exists a connected open subset $\hat{O} \subset \mathbb{R}^d$ and a compact connected set $U \Subset \hat{O}$ such that $V > \lambda_0$ in $\hat{O} \backslash U$, and $V < \lambda_0$ in $U \cup \mathbb{R}^d \backslash \hat{O}$.
- V is holomorphic in $\{\tilde{x} \in \mathbb{C}^d \mid |\operatorname{Im} \tilde{x}| \leqslant \varepsilon_0 |\operatorname{Re} \tilde{x}|, \operatorname{Re} \tilde{x}$ in a neighborhood of $\mathbb{R}^d \backslash \hat{O}\}$,
- $\lim_{|t| \nearrow \infty} |\Pi(\phi^t(x,\xi))| = \infty$, for all $(x,\xi) \in \mathbb{R}^d \backslash \hat{O} \times \mathbb{R}^d$ such that $p(x,\xi) = \xi^2 + V(x) = \lambda_0$. ϕ^t is the Hamiltonian flow of p and $\Pi : T^\star \mathbb{R}^d \to \mathbb{R}^d$ the projection.
- There exists a family of open sets $\Omega(h) \in \mathbb{C}$, such that $\bigcap_{h>0} \Omega(h) = \{\lambda_0\}$, $I(h) := \Omega(h) \cap \mathbb{R} \neq \emptyset$, and such that $\Omega(h) \cap \Gamma(h)$ contains only one resonance that we note $r(h)$. Moreover we assume that there exist $\varepsilon > 0$ and $c > 0$ such that $\operatorname{dist}(\Gamma(h), \partial\Omega(h)) \geqslant c e^{-\varepsilon/h}$.

Under these assumptions and (0.11), we have the following result:

Theorem 0.6 (Breit–Wigner formula). *Let $\delta(h)$ a family of positive real numbers such that, $\lim_{h \searrow 0} h^{-d}\delta(h) = 0$, $\lim_{h \searrow 0} |\operatorname{Im} r(h)|^{-1}\delta(h) = +\infty$, and satisfying $\operatorname{Re} r(h) \pm \delta(h) \in I(h)$, for all h small enough. Then*

$$\lim_{h \searrow 0} s_p(\operatorname{Re} r(h) \pm \delta(h), h) - s_p(\operatorname{Re} r(h), h) = \pm \tfrac{1}{2}.$$

1. Fredholm determinants and spectral distributions

We are going to use the Schatten ideals $\mathbf{S_p}$ (on $L^2(\mathbb{R}^d)$) thus we recall some basic definitions and properties. For more details see [19,37,38]. An operator A belongs to $\mathbf{S_p}$ if $|A| = (A^\star A)^{1/2}$ is compact with a spectrum in $l^p(\mathbb{N})$. If this holds true, then A is compact too and its spectrum $(\lambda_j)_{j\in\mathbb{N}} \in l^p(\mathbb{N})$, thus we can define

$$\mathrm{Det}_p\,(1+A) = \prod_{j=0}^{\infty}\,(1+\lambda_j)\exp\left(\sum_{k=1}^{p-1}\frac{(-1)^k}{k}\lambda_j^k\right).$$

If $p=1$, $\mathbf{S_1}$ is the set trace-class operators. An important property of Schatten classes that we shall use is the following: if $A_1,\dots,A_p\in\mathbf{S_p}$, then $A_1A_2\cdots A_p\in\mathbf{S_1}$. Let us finally recall that the pseudo-differential operator $\langle x\rangle^{-\varrho}\langle D\rangle^{-\varrho}$ belongs to $\mathbf{S_p}$ if $\varrho>d/p$. This justifies the relation between ρ and the class $\mathbf{S_p}$ used in the introduction and our results.

1.1. The Stieltjes transform

For any $u\in\varepsilon'(\mathbb{R})$, the space of compactly supported distributions, one can define the so-called *Stieltjes transform* of u as the following holomorphic function on $\mathbb{C}\backslash\mathrm{supp}(u)$

$$U(z) = \langle u, (.-z)^{-1}\rangle,$$

and one can recover u from U thanks to the inversion formula:

$$\lim_{\varepsilon\searrow+0} U(\lambda+i\varepsilon) - U(\lambda-i\varepsilon) = 2i\pi u(\lambda), \quad \text{in } \mathscr{D}'(\mathbb{R}).$$

This is essentially equivalent to the fact that $2i\pi\delta_0(\lambda) = (\lambda-i0)^{-1}-(\lambda+i0)^{-1}$ and very elementary. If one considers temperate distributions in $\mathscr{S}'(\mathbb{R})$, one cannot define the Stieltjes transform in general. However we describe below some sufficient conditions (at infinity) on $u\in\mathscr{S}'(\mathbb{R})$ ensuring the existence of the Stieltjes transform for temperate distributions.

If u is temperate, the following formula makes sense:

$$U_s(z) = \langle u, (.-z)^{-s}\rangle \tag{1.1}$$

for $\mathrm{Re}\, s > x_0 \gg 1$ and $z\notin\mathbb{R}$ (or $z\in\mathbb{C}\backslash[\inf\mathrm{supp}(u),\infty)$ if the inf is finite). Since we have

$$(\lambda-z)^{-1} = \partial_z(\partial_s(\lambda-z)^{-s})_{|s=0} \tag{1.2}$$

with $z\notin[\lambda,\infty)$, it is natural to consider the existence of a meromorphic continuation at $s=0$ for $U_s(z)$. Note that the complex powers are defined on $\mathbb{C}\backslash[0,\infty)$ with the principal determination of the log.

We give now sufficient conditions to get such a continuation. Assume that $u\in\mathscr{S}'(\mathbb{R})$, with $\inf(\mathrm{supp}(u)) > -\infty$, and that its Laplace transform, defined as

$Lu(t) = \langle u, f_t \rangle$ (with $f_t(\lambda) = e^{-t\lambda}$) has the following finite expansion

$$Lu(t) = \sum_{j=0}^{n} \sum_{k=0}^{k(j)} a_{jk} t^{\alpha_j} \log^k t + \mathcal{O}(t^{\beta_n}), \quad t \searrow 0^+, \tag{1.3}$$

where the a_{jk} are complex numbers, $\alpha_0 < \alpha_1 < \cdots < \alpha_n$ are real numbers and $\beta_n > 0$. Moreover we assume as in [29] that

$$-\alpha_j \in \mathbb{N} \Rightarrow a_{jk} = 0 \quad \forall k > 0. \tag{1.4}$$

Notice that these conditions hold if $u \in \mathscr{E}'(\mathbb{R})$ since, in that case, Lu is an entire function of t. Under the assumptions (1.3) and (1.4) we have

Proposition 1.1. (i) *The map $s \mapsto U_s(z)$ has a meromorphic continuation to the half plane* Re $s > -\beta_n$. *This continuation is regular at* 0 *and is holomorphic with respect to* $z \in \mathbb{C} \backslash [\inf \operatorname{supp}(u), \infty)$.

(ii) *If we set* $U(z) := \partial_z \partial_s U_s(z)_{|_{s=0}}$, *the following inversion formula holds*:

$$\lim_{\varepsilon \searrow 0} U(\lambda + i\varepsilon) - U(\lambda - i\varepsilon) = 2i\pi u(\lambda), \quad \textit{in } \mathscr{D}'(\mathbb{R}).$$

We shall call the function U, defined in this proposition, the Stieltjes transform of u. Formula (1.2) shows that it extends obviously the definition valid on $\mathscr{E}'(\mathbb{R})$.

Proof. (i) The result is local with respect to z so we fix $r > 0$ and assume that $\operatorname{Re} z < r$. If $\chi \in C_0^\infty$ is identically 1 near $(-r-1, r+1)$, the proposition is true for χu, therefore we may assume that u is supported in $[r+1, \infty)$, since replacing u by $(1-\chi)u$ does not change the assumptions. Then we use the fact that $(\lambda - z)^{-s} = \Gamma(s)^{-1} \int_0^\infty e^{-t(\lambda - z)} t^{s-1}\, dt$ when $\operatorname{Re} s > 0$ and $\lambda > r$ to write

$$U_s(z) = \frac{1}{\Gamma(s)} \int_0^\infty Lu(t) e^{tz} t^{s-1}\, dt, \quad \operatorname{Re} s > x_0$$

since $Lu(t) = \mathcal{O}(e^{-(r'+1)t})$ when $t \to \infty$, for any $r' < r$. Using $\int_0^1 t^\nu \log^k t\, dt = (-1)^k k!/(\nu+1)^{k+1}$ combined with (1.3), we get

$$\int_0^1 Lu(t) e^{tz} t^{s-1}\, dt = \sum_{j=0}^{n} \sum_{k=0}^{k(j)} \sum_{l=0}^{\infty} a_{jk} \frac{z^l}{l!} \frac{(-1)^k k!}{(s + \alpha_j + l)^{k+1}} + R_n(z, s)$$

with $R_n(z,s)$ holomorphic with respect to s when $\operatorname{Re} s > -\beta_n$, and holomorphic with respect to z. This proves the existence of the meromorphic continuation. The other integral, $\int_1^\infty$ does not cause any problem since it is entire with respect to s. Then, using condition (1.4) and the fact that $\Gamma(s)^{-1}$ vanishes at 0 we get the regularity at $s = 0$.

(ii) By the same reduction as for (i), we may assume that $\operatorname{Re} z < r$ and that u is supported in $[r+1, +\infty)$; in that case we know, thanks to the proof of (i), that $U_s(z)$ is holomorphic on $V(0) \times \{\operatorname{Re} z < r\}$, where $s \in V(0)$ a neighborhood of 0. Hence $\partial_z \partial_s U(0,z) - \partial_z \partial_s U(0,\bar{z}) \to 0$ when $\operatorname{Im}(z) \searrow 0$, locally uniformly with respect to $\operatorname{Re} z$, and this completes the proof. $\square$

Remark that, if we replace assumption (1.3) by the following:

$$Lu(t) \sim \sum_{j=0}^{\infty} \sum_{k=0}^{k(j)} a_{jk} t^{\alpha_j} \log^k t, \quad t \searrow 0^+, \tag{1.5}$$

with $\alpha_j < \alpha_{j+1}$ and $\alpha_j \to \infty$, we get a meromorphic continuation on $\mathbb{C}$ and the poles are the points

$$s_{j,l} = -\alpha_j - l, \qquad \alpha_j + l \notin \mathbb{N},$$

where $j, l \in \mathbb{N}$. Moreover, if for all j and l, $\alpha_j + l \notin \mathbb{N}$, then the meromorphic continuation of $U_s(z)$ vanishes at any s such that $-s \in \mathbb{N}$.

Now we turn to an example of computation of Stieltjes transform. Consider a family of self-adjoint operators $H_\varepsilon = H_0 + \varepsilon V$, depending on $\varepsilon \in [0,1]$. Assume moreover that H_0 is semibounded from below and that V is H_0 compact; more precisely, we assume that, for one $z \notin \sigma(H_0)$ (and hence for all) we have

$$V(H_0 - z)^{-\nu} \in \mathbf{S_p}, \quad \text{for some } \nu \in [0,1).$$

Then the domain of H_ε is independent of ε, $H_\varepsilon \geqslant \min(\inf \sigma(H_0), \inf \sigma(H_0 + V))$ and one can consider the functional

$$\langle u, f \rangle = \operatorname{Tr}\Big(f(H_0 + V) - \sum_{j=0}^{p-1} \frac{1}{j!} \frac{d^j}{d\varepsilon^j} f(H_\varepsilon)_{|_{\varepsilon=0}}\Big). \tag{1.6}$$

In order to show that (1.6) makes sense for any test function f we proceed as follows. The right-hand side of (1.6) vanishes if f is supported in $(-\infty, \min(\inf \sigma(H_0), \inf \sigma(H_0 + V)))$ and is well defined if $f = f_z$ with $f_z(\lambda) = (\lambda - z)^{-1}$. In that case, we have

$$\langle u, f_z \rangle = (-1)^p \operatorname{Tr}((H_0 + V - z)^{-1} (V(H_0 - z)^{-1})^p),$$

since $\partial_\varepsilon^j R_\varepsilon(z) = j!(-1)^j R_\varepsilon(z)(V R_\varepsilon(z))^j$. Using the Cauchy formula, we can define $\langle u, f \rangle$ for $f(\lambda) = (\lambda - E)^{-s}$, $\operatorname{Re} s > 1$ and E real, below the spectra. Then, by the Mellin transform, we can extend u to the class of Schwartz functions supported in (E, ∞). Thus u defines a temperate distribution. Using these considerations, we can define $U_s(z)$ as in (1.1) and we get the following result.

Proposition 1.2. *For all complex number* $z \notin [\min(\inf \sigma(H_0), \inf \sigma(H_0 + V)), \infty)$, $s \mapsto U_s(z)$ *is holomorphic in the region* $\operatorname{Re} s > -p(1-v)$ *and we have*

$$\partial_s U_s(z)_{|_{s=0}} = -\log \operatorname{Det}_p(1 + VR_0(z)),$$

where the condition $\log \operatorname{Det}_p(1 + VR_0(z)) \to 0$ *as* $\operatorname{Re} z - |\operatorname{Im} z| \to -\infty$ *determines the branch of the logarithm.*

Proof. Assume that $z \in K$, with K compact. Thus we can consider a contour $\mathscr{C}_K$, surrounding $[\min(\inf \sigma(H_0), \inf \sigma(H_0 + V)), \infty)$, such that $k - z \notin (-\infty, 0]$, for all $k \in \mathscr{C}_K$ and $z \in K$ (roughly speaking, the spectrum lies inside the contour and K lies outside). By the Cauchy formula, we have

$$(H_\varepsilon - z)^{-s} = \frac{i}{2\pi} \int_{\mathscr{C}_K} (k - z)^{-s} R_\varepsilon(k)\, dk. \tag{1.7}$$

Notice that we can choose $\mathscr{C}_K$ such that, outside a compact set, it coincides with the union of the two half lines $e^{\pm i\pi/4}[1, \infty)$ so that $R_\varepsilon(k) = \mathcal{O}(|k|^{-1})$ at infinity. Then we get easily that

$$U_s(z) = (-1)^p \operatorname{Tr}\left(\frac{i}{2\pi} \int_{\mathscr{C}_K} (k - z)^{-s} R(k)(VR_0(k))^p\, dk\right). \tag{1.8}$$

Since the trace norm of $R(k)(VR_0(k))^p$ is $\mathcal{O}(|k|^{-1-p(1-v)})$, the right-hand side is holomorphic with respect to s when $\operatorname{Re} s > -p(1-v)$, thus it is regular at $s = 0$. Then if we differentiate with respect to z, we get, thanks to formula (1.2)

$$\partial_z \partial_s U_s(z)_{|_{s=0}} = (-1)^p \operatorname{Tr}\left(R(z)(VR_0(z))^p\right)$$

whose right-hand side is exactly the logarithmic derivative of $\operatorname{Det}_p(1 + VR_0(z))$, up to the sign (see for example [38]). Therefore, we just have to show that $\partial_s U_s(z)_{|_{s=0}} \to 0$ when $z \to -\infty$. If z is real and $z \to -\infty$, we can let the contour be $\mathscr{C}_z = \bigcup_\pm e^{\pm i\pi/4}[z + 1, \infty)$; by change of variable we get

$$\partial_s U_s(z)_{|_{s=0}} = (-1)^p \operatorname{Tr}\left(\frac{i}{2\pi} \int_{\mathscr{C}_{-1/2}} -\log(k - c) R(k + z - c)(VR_0(k + z - c))^p\, dk\right),$$

where $c = \min(\inf \sigma(H_0), \inf \sigma(H_0 + V)) + 1/2$. The limit of the right-hand side is 0 since the trace norm of $R(k + z - c)(VR_0(k + z - c))^p$ is still $\mathcal{O}(|k|^{-1-p(1-v)})$ uniformly with respect to z. The proof is complete. □

Proposition 1.2 was proved by Koplienko in [25] under the condition $V \in \mathbf{S_p}$, and is essentially well known. We have given the proof in the more general case $V(H_0 - z)^{-v} \in \mathbf{S_p}$ for the sake of completeness, and to justify the remark after the proof theorem (0.2) (see the next subsection).

1.2. Proof of Theorem 0.2

Consider $H_0 = p(x, D)$ and $H_0 + V = p(x, D) + v(x, D)$ two uniformly elliptic differential operators, where $p(x, \xi) = \sum_{|\alpha| \leqslant 2m} p_\alpha(x)\xi^\alpha$, and $v(x, \xi) = \sum_{|\alpha| \leqslant 2m} v_\alpha(x)\xi^\alpha$, with p_α and v_α smooth functions such that for all $|\alpha| \leqslant 2m$ and $\beta \in \mathbb{N}^d$

$$|\partial^\beta p_\alpha(x)| \leqslant C_\beta, \quad |\partial^\beta v_\alpha(x)| \leqslant C_\beta \langle x \rangle^{-\rho}, \quad x \in \mathbb{R}^d.$$

Their order is the integer $2m$ which is even since we do the following ellipticity assumption

$$\sum_{|\alpha|=2m} p_\alpha(x)\xi^\alpha \geqslant C|\xi|^{2m}, \quad \sum_{|\alpha|=2m} (p_\alpha + v_\alpha)(x)\xi^\alpha \geqslant C|\xi|^{2m}, \quad x, \xi \in \mathbb{R}^d$$

for some constant $C > 0$. We also assume that these operators are symmetric on L^2, which allows us to consider their self-adjoint realization with domain $H^{2m}(\mathbb{R}^d)$, the usual Sobolev space. These operators are clearly semibounded from below.

Now, if $\rho > d/p$ and u_p is the spectral distribution associated to $(H_0, H_0 + V)$, we know by Bouclet [4,7] that

$$\langle u_p, e^{-t(.)} \rangle \sim t^{-d/2m} \sum_{k \geqslant 0} a_k t^{k/m}, \quad t \searrow 0,$$

therefore we can apply Proposition 1.1 and we get the theorem.

Remark 1.3. If V is of order $m' < 2m$, it is H_0-compact and if $2m - m' > d/p$, $V(H_0 + i)^{-1} \in \mathbf{S_p}$. Furthermore, for some $\nu \in [0, 1)$ we still have $\nu 2m - m' > d/p$ and thus $V(H_0 + i)^{-\nu} \in \mathbf{S_p}$ which allows us to use Proposition 1.2; this proves that $D_p(z)$ coincides, in this case, with the usual determinant $\mathrm{Det}_p(1 + V(H_0 - z)^{-1})$.

2. The asymptotic representation formula

In this section we prove an asymptotic formula for $u_p(\lambda, h)$ for positive and noncritical energies of the free Hamiltonian H_0. For this technical result, u_p is associated to the same kind of pseudo-differential operators, $H_0, H_0 + V$, as the one considered in [7,34]. This means that

$$H_0 = \omega(hD),$$

with $\omega(\xi) \to +\infty$ as $\xi \to \infty$, and $\omega \geqslant 0$, with ω is smooth and temperate, in the sense that, for some C and M positive numbers, $\omega(\xi) \leqslant C(1 + \omega(\eta))\langle \xi - \eta \rangle^M$, for all ξ, η.

Then we consider a perturbation

$$\begin{aligned} V &= q_0(x,hD) + \cdots + h^d q_d(x,hD) + h^{d+1}\tilde{q}(x,hD,h), \\ &= q(x,hD,h) + h^{d+1}\tilde{q}(x,hD,h). \end{aligned}$$

We shall explain further on why we split V into these two pieces. Recall that if a is a symbol, $a^w(x,hD)$ is the operator defined by

$$a^w(x,hD)u(x) = (2\pi h)^{-d} \int \int e^{i\langle x-y,\xi\rangle} a\Big(\frac{x+y}{2},\xi\Big)u(y)\,dy\,d\xi, \quad u\in\mathscr{S}(\mathbb{R}^d).$$

We assume that $h^{d+1}\tilde{q}_d \sim \sum_{k\geqslant d+1} h^k q_k$, which means as usual that $h^{d+1}\tilde{q}_d - \sum_{d+1\leqslant k<N} h^k q_k = \mathcal{O}(h^{d+1+N})$ in $S_1(\omega,-N)$. We also assume that for all $k\in\mathbb{N}$

$$q_k \in S_1(\omega,-k)\cap S_1(\omega,-\rho), \quad \rho>0,$$

with $\rho>0$ fixed. This is the *long-range* condition on the perturbation V. The symbol class $S_1(\omega,\mu)$ is the space of symbols a such that the semi-norms

$$\sup_{\mathbb{R}^{2d}} \max_{|\alpha+\beta|\leqslant N} |\langle x\rangle^{-\mu+k}(1+\omega(\xi))^{-1}\partial_x^\alpha\partial_\xi^\beta a(x,\xi)|$$

are finite for all N. We shall sometimes need the class $S_1(\omega^n,\mu)$ which the same as above except that ω has to be replaced by ω^n. We will also use $S_1(-\infty,\mu) = \bigcap_{n>0} S_1(\Omega^n,\mu)$ where $\Omega(\xi) = \langle\xi\rangle^{-1}$.

Finally we assume that q and $\tilde{q}$ are real valued and that, for some constant $C>0$

$$C + w(\xi) + q_0(x,\xi) > C^{-1}(1+\omega(\xi)), \quad \forall x,\xi\in\mathbb{R}^d. \tag{2.1}$$

These conditions entail the essential self-adjointness of $H_0+\varepsilon V$ for all $\varepsilon\in[0,1]$ and $h\in(0,h_0]$ (h_0 small enough); the domain of the self-adjoint realization is independent of ε (see [7]).

Let $I\subset(0,\infty)$ be a noncritical interval for ω. Our main result is the following:

Theorem 2.1. *Let $p\in\mathbb{N}$ be such that $\rho>d/p$. There exists $\chi\in C_0^\infty(\mathbb{R}^d)$, and a family of functions $u_{\mathrm{reg},p}(.,h)\in C^\infty(I)$, such that for all $J\Subset I$ one can find sequences of operators $\Upsilon_j^\pm(\lambda,h)$, $j\geqslant 0$, such that for all $N>0$*

$$u_p(\lambda,h) = \mathrm{Tr}\Big(\chi\frac{\partial E}{\partial\lambda}(\lambda,h)\chi\Big) + u_{\mathrm{reg},p}(\lambda,h) + h^{k(N)}\mathscr{R}_N(\lambda,h), \quad h\searrow 0 \tag{2.2}$$

$$\mathscr{R}_N(\lambda,h) = \mathrm{Tr}(\Upsilon_N^+(\lambda,h)R(\lambda+i0,h)) + \mathrm{Tr}(\Upsilon_N^-(\lambda,h)R(\lambda-i0,h)), \tag{2.3}$$

where $k(N)\uparrow+\infty$ as $N\uparrow\infty$. $E(\lambda,h)$ is the spectral resolution of H_0+V and $R(z,h)$ its resolvent. The function $u_{\mathrm{reg},p}(.,h)$ has a full asymptotic expansion in $C^\infty(J)$; this *means that there exists a sequence of functions* $u^j_{\mathrm{reg},p} \in C^\infty(J)$ *such that for all N*

$$u_{\mathrm{reg},p}(.,h) - h^{-d} \sum_{j=0}^{N} h^j\, u^j_{\mathrm{reg},p} = \mathcal{O}(h^{N-d}) \quad \textit{in the } C^\infty(J) \textit{ topology.}$$

The operators $\Upsilon^{\pm}_N(\lambda,h)$ *are such that*

$$\langle x \rangle^{k(N)} \Upsilon^{\pm}_N(.,h) \langle x \rangle^{k(N)} \textit{ is bounded in } C^{k(N)}(J, \mathbf{S_1}), \quad \textit{as } h \searrow 0.$$

By the same arguments as those of [4,7,34], Theorems 0.4, 0.5 and 0.6 are simple corollaries of Theorem 2.1 and the rest of this section is devoted to its proof.

Remark that our assumptions on the operators are satisfied by the long-range Schrödinger operator ($\omega(\xi) = \xi^2$, $\tilde{q}(x,\xi,h) = 0$, $q(x,\xi,h) = q_0(x,\xi) = V(x)$)

$$H_0 = -h^2 \Delta, \quad V = V(x)$$

and by the rescaled elliptic operators of Section 1.2

$$H_0 = h^{2m} p_0(D), \quad V = h^{2m} \sum_{|\alpha| \leqslant 2m} v_\alpha(x) D^\alpha.$$

This is useful to prove high-energy asymptotics by considering these operators near the rescaled energy level $\lambda' = 1$ and by choosing $h^{2m} = \lambda^{-1}$, with $\lambda \nearrow \infty$, since one sees easily that

$$h^{2m} u_p(\lambda', h) = u_p(\lambda'/h^{2m})$$

if $u_p(.,h)$ is associated with the above pair $H_0, H_0 + V$ and $u_p(.) = u_p(.,1)$.

The main tool of the proof is the Isozaki–Kitada parametrix (see [16,24,34]) for the operators $H_\varepsilon = H_0 + \varepsilon V$, $\varepsilon \in [0,1]$ and especially the study of the differentiability of this parametrix with respect to ε. For the sake of completeness as well as to be able to state precise results we have chosen to write Section 3 on this subject.

In the proof of Theorem 2.1, we are going to approximate the symbol of V by a fast decaying one (w.r.t. x) in order to use the SSF. To that end we choose a subset

$$\mathscr{B} \subset S_1(\omega, -\mu), \quad \mu > d$$

in which all the symbols are real valued and satisfy (2.1). We can assume that $\mathscr{B}$ is bounded in $S_1(\omega, -\rho)$ and that it contains a sequence converging to q in $S_1(\omega, -\rho_1)$ for any $\rho_1 < \rho$. Since we only assume that $\rho > d/p$, there is no loss of generality if we assume that $\rho = \rho_1$ and that $\rho^{-1} \notin \mathbb{N}$ (this purely technical condition turns up in Appendix A).

Finally let us mention that in our constructions, $q = q(x,\xi,h)$ will play the role of the principal symbol. It is more convenient, for technical reasons, to do so since $\tilde{q}(h)$ is integrable with respect to x which is useful when one wants to use trace class operators.

2.1. Two technical lemmas

Let $\chi \in C_0^\infty(\mathbb{R}^d)$ be a cutoff function such that $\chi \equiv 1$ on the ball of radius $R>0$.

Lemma 2.2. *One can choose $R>0$ large enough such that, for all functions $f_1, f_2 \in C_0^\infty(\mathbb{R})$ satisfying $f_1 \equiv 1$ near $\operatorname{supp} f_2$, we have the following estimates in $\mathbf{S_1}$:*

$$\langle x \rangle^N (1-\chi)(1-f_1(H_0)) f_2(H_0+\varepsilon V) \langle x \rangle^N = \mathcal{O}(h^N), \quad \forall N.$$

These estimates are uniform with respect to $\varepsilon \in [0,1]$ and to the principal symbol of V in $\mathscr{B}$. Moreover, for all $j \geqslant 0$, the following operator is pseudo-differential

$$\frac{d^j}{d\varepsilon^j}((1-\chi)(1-f_1(H_0)) f_2(H_0+\varepsilon V))_{|_{\varepsilon=0}}.$$

Its symbol is $\mathcal{O}(h^\infty)$ in $S_1(1,-\infty)$ and depends continuously on q in $\mathscr{B}$.

Proof. The first statement is a simple consequence of the fact that, for all N large enough

$$f_2(H_0+\varepsilon V) = a_{N,\varepsilon}(x, hD, h) + h^N R_{N,\varepsilon}(h)$$

with $|||\langle x \rangle^{N/4} R_{N,\varepsilon} \langle x \rangle^{N/4}|||_1 = \mathcal{O}(h^{N/4})$ uniformly with respect to ε and $q \in \mathscr{B}$. The symbol $a_{N,\varepsilon}(x,\xi,h)$ is a linear combination of $a_n(x,\xi) f_2^{(n)}(\omega(\xi)+\varepsilon q(x,\xi))$ with $a_n \in S_1(\omega^{k(n)}, -n)$, for some $k(n) \geqslant 0$. If x is large enough, and $\omega(\xi)+\varepsilon q(x,\xi)$ lies in $\operatorname{supp} f_2$, then $\omega(\xi) f_1^{-1}(1)$, and then, the composition formula of pseudo-differential operators yields

$$(1-\chi)(1-f(H_0)) a_{N,f,\varepsilon}(x, hD, h) \in h^\infty Op(\mathscr{S}),$$

where $\mathscr{S}$ is the Schwartz class. The second Part is a direct consequence of the fact that $\partial_\varepsilon^j f(H_0+\varepsilon V)_{|_{\varepsilon=0}}$ is purely pseudo-differential. $\square$

In all what follows, I is an open interval and we use the notation $U_\varepsilon(t) = \exp(-itH_\varepsilon/h)$. Consider a family of bounded operators B_ε which is C^p with respect to $\varepsilon \in [0,1]$ and such that $B_\varepsilon \langle x \rangle^{\bar{d}}$ is still C^p in the class of bounded operators for some $\bar{d} > d$. Then we have

Lemma 2.3. *The family of operators $B_\varepsilon f(H_\varepsilon) U_\varepsilon(t)$ is smooth in the trace class (w.r.t. $\varepsilon \in [0,1]$) for any $f \in C_0^\infty(I)$. Moreover for all $j \geqslant 1$*

$$\frac{d^j}{d\varepsilon^j} \operatorname{Tr}(B_\varepsilon f(H_\varepsilon) U_\varepsilon(t))_{|_{\varepsilon=0}}$$

can be written as a linear combination of the traces of the following operators with $0 \leqslant n \leqslant j$:

$$\frac{\partial_\varepsilon^{j-n} B_\varepsilon}{h^n}\bigg|_{\varepsilon=0} \int_0^t \int_0^{t-t_1} \cdots \int_0^{t-t_1-\cdots-t_{n-1}} a_0(x, hD, h) \\ \times \prod_{k=0}^{n} U_0(t_k - t_{k+1}) a_{k+1}(x, hD, h)\, dt_n \cdots dt_2\, dt_1.$$

Here the product is the composition from the left to the right (*with the term* $k = 0$ *on the left*). *Moreover we have used the following conventions*:

$$t_0 = t, \quad t_{n+1} = 0$$

and if $n = 0$ *there is no integral sign. The symbols* a_l *belong to* $S_1(-\infty, -\rho)$ *if* $1 \leqslant l \leqslant n$, *whereas* a_0 *and* a_{n+1} *are in* $S_1(-\infty, 0)$. *They all have an expansion with respect to h and are compactly supported with respect to* ξ *in a small neighborhood of* $\omega^{-1}(\operatorname{supp} f)$. *As in the previous lemma, everything depends continuously on* $q \in \mathscr{B}$.

Proof. This lemma is a simple consequence of

$$\frac{d}{d\varepsilon} U_\varepsilon(t) = -\frac{i}{h} \int_0^t U_\varepsilon(t - t_1) V U_\varepsilon(t_1)\, dt_1, \tag{2.4}$$

together with the fact that for all n, $(d/d\varepsilon)^n f(H_\varepsilon)|_{\varepsilon=0}$ is pseudo-differential operator with symbol in $S_1(1, -n\rho)$ (see [7]). Moreover, using the fact that $(d/d\varepsilon)(f(H_\varepsilon)U_\varepsilon(t))$ can always be written as

$$\left(\frac{d}{d\varepsilon} f(H_\varepsilon)\right) U_\varepsilon(t) f_1(H_\varepsilon) + f(H_\varepsilon)\left(\frac{d}{d\varepsilon} U_\varepsilon(t)\right) f_1(H_\varepsilon) + f(H_\varepsilon) U_\varepsilon(t) \left(\frac{d}{d\varepsilon} f_1(H_\varepsilon)\right)$$

with $f_1 f = f$, we see easily that the symbols a_n are supported in any arbitrary small neighborhood of $\omega^{-1}(\operatorname{supp} f)$. □

2.2. A nonstationary phase lemma

In the proof of Theorem 2.1 we shall need to deal with Fourier transforms of traces of operators similar to the ones of Lemma 2.3. We want to show that these traces are smooth with respect to the spectral parameter hence we need to prove that the Fourier transforms decay fast with respect to the time parameter. This will be achieved by mean of the lemma which we prove in this subsection.

Let us consider the set of all complex-valued functions f such that, for all

$$\alpha_2, \ldots, \alpha_k \in \mathbb{N}^d, \quad \beta_1, \ldots, \beta_{k-1} \in \mathbb{N}^d, \quad \beta_k \in \mathbb{N}^d, \quad |\beta_k| \leqslant N$$

one can find a constant C such that

$$|\partial^{\alpha_2\cdots\alpha_k}_{x_2\cdots x_k}\partial^{\beta_1\cdots\beta_{k-1}}_{\xi_1\cdots\xi_{k-1}}\partial^{\beta_k}_{\xi_k} f(x_0,x_1,\ldots,x_k,\xi_1,\ldots,\xi_k,h)|\leqslant Ch^{-|\beta_k|}\langle x_0,x_1\rangle^{|\beta_k|}$$
$$\times\langle x_0\rangle^{-N}\langle x_{0,1}\rangle^{-\delta_1}\cdots\langle x_{0,n}\rangle^{-\delta_1}\langle x_1\rangle^{-N-\delta}\langle x_2\rangle^{-\delta-|\alpha_2|}\cdots\langle x_k\rangle^{-\delta-|\alpha_k|} \quad (2.5)$$

with $\delta_1>1$, $\delta=d\delta_1$, and where the variables are such that

$$x_0=(x_{0,1},\ldots,x_{0,n})\in\mathbb{R}^n,\quad x_1,\ldots,x_k,\ \ \xi_1,\ldots,\xi_k\in\mathbb{R}^d,\ \ h\in(0,1].$$

One also assume that the function f are supported in $(\omega^{-1}(I))^k$ with respect to $\xi_1,\ldots,\xi_k$. The best constants C in (2.5) are semi-norms and define the topology on the set.

Remark. 1. The set of variables x_0 can be empty, and that will be actually the case in the applications. However, it is more convenient for the proof of the lemma below to consider such functions.

2. The functions f introduced above look very much like the symbols of the oscillatory integrals given by the kernels of the operators involved in Lemma 2.3.

Our nonstationary phase result is the following:

Lemma 2.4. *There exists a constant $C>0$ which depends on a finite number of semi-norms of f such that*

$$\left|\int_{(\mathbb{R}^d)^k\times(\mathbb{R}^d)^k} e^{\frac{i}{h}\phi_k} f dx_1\cdots dx_k\, d\xi_1\cdots d\xi_k\right|\leqslant C\langle x_{0,1}\rangle^{-\delta_1}\cdots\langle x_{0,n}^{-\delta_1}\rangle\left(\frac{h}{t}\right)^N \quad (2.6)$$

for all $t\neq 0$ and $h\in(0,1]$.

Proof. We proceed by induction on $k\geqslant 2$. We shall do integrations by parts using the following differential operators, which all leave $\exp(i\phi_k/h)$ invariant,

$$P_l=-\frac{h}{i}\frac{\sum_{j=1}^l t_j\nabla\omega(\xi_j)}{|\sum_{j=1}^l t_j\nabla\omega(\xi_j)|^2}\left(\sum_{j=1}^l\nabla_{\xi_j}\right)-(x_{l+1}-x_1)\frac{\sum_{j=1}^l t_j\nabla\omega(\xi_j)}{|\sum_{j=1}^l t_j\nabla\omega(\xi_j)|^2},\quad 1\leqslant l\leqslant k,$$

where one should notice that if $l=k$ then $x_1-x_{l+1}=0$. We shall use as well

$$Q_l=\frac{h}{i}\frac{\xi_l-\xi_{l-1}}{|\xi_l-\xi_{l-1}|^2}\nabla_{x_l}.$$

Let us start with $k=2$. We choose $\chi\in C_0^\infty(\mathbb{R}^d)$, $\chi\equiv 1$ near 0, and we write $f=f_1+f_2$, with $f_1=\chi(\xi_1-\xi_2)f$. By choosing the support of χ small enough, and using the fact that $|\nabla\omega(\xi)|\geqslant\varepsilon>0$ on $\omega^{-1}(I)$, we get

$$|t_1\nabla\omega(\xi_1)+t_2\nabla\omega(\xi_2)|\geqslant\frac{\varepsilon}{2}t$$

on the support of f_1, and we do N integrations by parts with P_2 (which is independent of x_1, x_2 since $l = k$). This proves that the estimate (2.6) holds for f_1.

To estimate the same integral with f_2, we first do N integrations by parts with Q_2 which gives a gain of $h^N \langle x_2 \rangle^{-N}$ (and also a lost of $\langle x_1, x_2 \rangle^N$). Then:

- either $t_1 \geqslant t/2$, and we do N integrations by parts with P_1, which gives a gain of $t_1^{-N} = t^{-N}$,
- or $t_2 \geqslant t/2$, and we do N integrations by part with $\tilde{P}_1$, which is the same operator as P_1 but where we swap x_1 and x_2, and ξ_1 and ξ_2. We get a gain of $t_2^{-N} = t^{-N}$.

Thus the result is proved when $k = 2$.

Now assume that $k \geqslant 3$ and that the result holds to all orders $\leqslant k-1$. We define, for $1 \leqslant l \leqslant k-1$ the function $\chi_l = \chi(\xi_l - \xi_{l+1})$ and then write

$$f = \chi_1 \cdots \chi_{k-1} f + (1 - \chi_1) f + \sum_{l=2}^{k-1} \chi_1 \cdots \chi_{l-1}(1 - \chi_l) f.$$

By choosing the support of χ small enough (and $\chi \equiv 1$ near 0), we get $|\sum_{j=1}^{k} t_j \nabla \omega(\xi_j)| \geqslant ct$ on the support of $\chi_1 \cdots \chi_{k-1}$ for some $c > 0$; then doing N integrations by parts with P_k shows that

$$\left| \int e^{\frac{i}{h}\phi_k} \chi_1 \cdots \chi_{k-1} f \, dx_1 \cdots dx_k \, d\xi_1 \cdots d\xi_k \right| \leqslant C \langle x_{0,1} \rangle^{-\delta_1} \cdots \langle x_{0,n} \rangle^{-\delta_1} \left(\frac{h}{t} \right)^N.$$

Now we consider the integral with $(1 - \chi_1) f$ in which we can do N integrations by parts with Q_2, from which we get a gain of $h^N \langle x_2 \rangle^{-N}$. There are again two cases:

- either $t_1 \geqslant t/2$, and we do exactly as for $k = 2$,
- or $t_2 + \cdots + t_k \geqslant t/2$, and we use the induction assumption. Precisely, the function that we integrate has the following form:

  $$e^{\frac{i}{h}\tilde{\phi}_{k-1}} \tilde{f}(\tilde{x}_0, \tilde{x}_2, \cdots, \tilde{x}_k, \tilde{\xi}_2, \tilde{\xi}_k),$$

  where the new variables are $\tilde{x}_0 = (x_0, x_1, \xi_1)$, $(\tilde{x}_j, \tilde{\xi}_j) = (x_j, \xi_j)$, when $2 \leqslant j \leqslant k$ and the new phase function is

  $$\tilde{\phi}_{k-1} = \sum_{j=2}^{k} \langle \tilde{x}_j - \tilde{x}_{j+1}, \xi_j \rangle - t_j \omega(\xi_j), \quad \tilde{x}_{k+1} = \tilde{x}_2.$$

  Note that $\phi_k - \tilde{\phi}_{k-1} = \langle x_1 - x_2, \xi_1 - \xi_k \rangle - t_1 \omega(\xi_1)$ so (2.5) holds for $\tilde{f}$ and we get

  $$\left| \int_{(\mathbb{R}^d)^{k-1} \times (\mathbb{R}^d)^{k-1}} e^{\frac{i}{h}\tilde{\phi}_{k-1}} \tilde{f} dx_2 \cdots dx_k d\xi_2 \cdots d\xi_k \right| \leqslant C \langle \tilde{x}_{0,1} \rangle^{-\delta_1} \cdots \langle \tilde{x}_{0,n+2d} \rangle^{-\delta_1} \left(\frac{h}{t} \right)^N$$

  from which we get the result after integration with respect to x_1, ξ_1.

Finally we have to consider the integrals involving $\chi_1 \cdots \chi_{l-1}(1-\chi_l)f$ with $2 \leqslant l \leqslant k-1$. Thanks to $(1-\chi_l)$, we can integrate by parts N times with Q_{l+1} from which we have a gain of $h^N \langle x_l \rangle^{-N}$ and there are again two possibilities:

- either $t_1 + \cdots + t_l \geqslant t/2$, and choosing the support of χ small enough we can integrate by parts N times with P_{l-1}. Then we get

$$\left| \int e^{\frac{i}{h}\phi_k} \chi_1 \cdots \chi_{l-1}(1-\chi_l) f dx_1 \cdots dx_l d\xi_1 \cdots d\xi_l \right| \leqslant C \left(\frac{h}{t}\right)^N \prod_{j=1}^{n} \langle x_{0,j} \rangle^{-\delta_1} \prod_{j=l+1}^{k} \langle x_j \rangle^{-\delta}$$

  from which the result is obvious,
- or $t_{l+1} + \cdots + t_k \geqslant t/2$. We use again the induction assumption, considering (x_0, x_1, ξ_1) as a new variable $\tilde{x}_0$ and $x_2, \xi_2, \ldots, x_{l-1}, \xi_{l-1}$ as parameters. We introduce the phase function

$$\tilde{\phi}_{k-l+1} = \sum_{j=l}^{k-1} \langle x_j - x_{j+1}, \xi_j \rangle - t_j \omega(\xi_j) + \langle x_k - x_l, \xi_k \rangle - t_k \omega(\xi_k),$$

  and we remark that $\phi_k - \tilde{\phi}_{k-l+1} - \langle x_l - x_1, \xi_k \rangle + \langle x_l, \xi_l \rangle$ depends only on $\tilde{x}_0$ and the parameters, then we get the result as before. The proof is now complete. $\square$

2.3. *The regular terms*

In this subsection we will study the distributions defined, for $\tilde{\chi} \in S_1(1, -\infty)$ and $j \geqslant 0$, by

$$\langle u(h), f \rangle = \mathrm{Tr}\left(\tilde{\chi}(x, hD) \frac{d^j}{d\varepsilon^j} f(H_\varepsilon)_{|_{\varepsilon=0}} \right), \quad f \in C_0^\infty(I).$$

It is already known that these distributions have a full asymptotic expansion in h in the weak sense (see [4,7]); this means that there exists a sequence $(u_k)_{k \in \mathbb{N}}$ of elements of $\mathscr{D}'(I)$, independent of h, such that for all $N \geqslant 0$

$$u(h) = h^{-d} \sum_{k=0}^{N} h^k u_k + h^{N+1-d} \tilde{u}_N(h), \quad h \searrow 0, \tag{2.7}$$

where $\tilde{u}_N(h)$ is a bounded family in $\mathscr{D}'(I)$ (i.e. $\langle \tilde{u}_N(h), f \rangle$ is bounded for all $f \in C_0^\infty(I)$). Our goal is to prove that the asymptotic (2.7) holds in the strong sense, that is in the C^∞ sense. This is the purpose of the following proposition. Recall that $I \Subset (0, \infty)$ is an open and noncritical interval for ω.

Proposition 2.5. *The distributions u_k are C^∞ on I and $\tilde{u}_N(h)$ is a bounded family in $C^\infty(I)$.*

Proof. By uniqueness of the asymptotic expansion, it is sufficient to show that $u(h)$ has a full asymptotic expansion in $C^\infty(J)$ for all $J \Subset I$. By choosing a cutoff function $g \in C_0^\infty(I)$, $g \equiv 1$ near J, we only have to study $gu(h)$, and we can consider its semiclassical Fourier transform which is

$$w(h,t) = \mathrm{Tr}\left(\tilde{\chi}(x, hD)\frac{d^j}{d\varepsilon^j}(g(H_\varepsilon)U_\varepsilon(t))\right)_{|_{\varepsilon=0}}.$$

By Lemma 2.3, $w(h,t)$ is a linear combination (with coefficients independent on h and t) of the following type of functions, with $0 \leqslant k \leqslant j$:

$$h^{-k}\int_0^t\int_0^{t_1}\cdots\int_0^{t_{k-1}}\mathrm{Tr}(\mathscr{K}_k(t-t_1, t_1-t_2, \dots, t_{k-1}-t_k, t_k, h))\,dt_k\cdots dt_2\,dt_1, \tag{2.8}$$

where, by convention, there is no $\int$ sign if $k=0$, and only $\int_0^t \cdots dt_1$ if $k=1$. Moreover we have

$$\begin{aligned}\mathscr{K}_k&(t_1, t_2, \dots, t_{k+1}, h)\\ &= a_1(x, hD, h)U_0(t_1)a_2(x, hD, h)U_0(t_2)\cdots a_{k+1}(x, hD, h)U_0(t_{k+1}),\end{aligned} \tag{2.9}$$

with symbols $a_1, \dots, a_{k+1}$ all supported in a small neighborhood of $\omega^{-1}(\mathrm{supp}\,(g))$ with respect to the ξ variable, having an asymptotic expansion with respect to h, and in the following classes:

$$a_1 \in \mathscr{S}(\mathbb{R}^{2d}), \quad a_j \in S_1(-\infty, -\rho), \quad 2 \leqslant j \leqslant k+1.$$

The Schwartz kernel $K_k(x_1, x_{k+2})$ of $\mathscr{K}_k(t_1, \dots, t_{k+1})$ is given by the oscillatory integral

$$\int e^{\frac{i}{h}\phi_{k+1}}a_1(x_1,\xi_1)a_2(x_2,\xi_2)\cdots a_{k+1}(x_{k+1},\xi_{k+1})dx_2\cdots dx_{k+1}d\xi_1\cdots d\xi_{k+1}, \tag{2.10}$$

where the phase function $\phi_{k+1} = \phi_{k+1}(x_1, \dots, x_{k+2}, \xi_1, \dots, \xi_{k+1})$ is given by

$$\phi_{k+1} = \sum_{j=1}^{k+1}\langle x_j - x_{j+1}, \xi_j\rangle - t_j\omega(\xi_j).$$

First we rewrite (2.10) as a convergent integral: using the decay of $K_k(x_1, x_{k+1})$ with respect to x_1, one can do kd integrations by parts in (2.10) with the operator $h|x_1 - x_2|^{-2}(x_1 - x_2).\nabla_{\xi_1}$, if $x_2 \neq x_1$ (if $x_1 - x_2$ is small the fast decay w.r.t. x_1 entails the fast decay w.r.t. x_2). Thus, by loosing $\mathcal{O}(t_1^{kd})$, we get a decay of $\langle x_2\rangle^{-kd-\rho}$. By repeating this operation with the other x_j variables, we can assume that the symbol is integrable with respect to each x_j since it decays like $\langle x_j\rangle^{-(k+2-j)d-\rho}$ with respect to each x_j. Notice that this generates a factor $\mathcal{O}(t^{k(k+1)d/2})$ which is armless since are now in position to use Lemma 2.4. It shows that $w(t,h) = \mathcal{O}((h/t)^\infty)$, if $|t| \geqslant c$ and

$h \in (0, h_0]$, thus, if $\varphi \in C_0^\infty(-2c, 2c)$ is such that $\varphi \equiv 1$ on $(-c, c)$, we have

$$\lambda \mapsto (2\pi h)^{-1} \int e^{\frac{i}{h}t\lambda}(1-\varphi)(t)w(t,h)dt = \mathcal{O}(h^\infty) \quad \text{in } C^\infty.$$

Now we consider the region $|t| \leqslant 2c$. The operator $\mathcal{K}_k$ in (2.8) is the product of $a_0(x, hD, h)U_0(t)$ and of the operators $U_0(-t_j)a_j(x, hD, h)U_0(t_j)$, $1 \leqslant j \leqslant k$, with $|t_j| \leqslant |t|$, which are, by the Egorov's theorem, pseudo-differential operators. Their product can be written $\tilde{a}_k(x, hD, t, k)$, with $\tilde{a}_k$ depending smoothly on t. Hence $\int \exp(it\lambda/h)w(t,h)\varphi(t)\, dt$ can be written as a linear combination of the

$$h^{-k} \int e^{\frac{i}{h}t\lambda} \mathrm{Tr}(a_0(x, hD, h)U_0(t)\tilde{a}_k(x, hD, t, h))\varphi(t)\, dt$$

which all have a full expansion in $C^\infty(J)$ (in increasing integer powers of h) thanks to the stationary phase theorem, and this completes the proof. □

2.4. *Proof of Theorem 2.1*

We have to investigate, for $\lambda \in J \Subset I$, the following distribution:

$$u_p(\lambda, h) = u_{1,1}(\lambda, h) - \sum_{j=0}^{p-1} \frac{1}{j!} \partial_\varepsilon^j u_{1,\varepsilon}(\lambda, h)|_{\varepsilon=0},$$

where, in the weak sense, we have

$$u_{1,\varepsilon}(\lambda, h) = \mathrm{Tr}\left(f_0(H_\varepsilon)\frac{\partial E_\varepsilon}{\partial \lambda} - f_0(H_0)\frac{\partial E_0}{\partial \lambda} \right),$$

with $f_0 \in C_0^\infty(I)$ and $f_0 \equiv 1$ near J. Actually, $u_{1,\varepsilon}$ is well defined, if the coefficients of the perturbation V decay as $\langle x \rangle^{-d-\varepsilon}$. This is why we approximate the symbol of V by a sequence of $\mathscr{B}$. Hence, let us assume that $q \in \mathscr{B}$.

Let $\chi \in C_0^\infty$ be a cutoff function such that $\chi \equiv 1$ on the ball of radius R. Thanks to the result of the previous subsection, we only have to consider

$$\mathrm{Tr}\left(f_0(H_\varepsilon)\frac{\partial E_\varepsilon}{\partial \lambda}(1-\chi^2) - f_0(H_0)\frac{\partial E_0}{\partial \lambda}(1-\chi^2) \right), \tag{2.11}$$

since the others terms are $\mathrm{Tr}(\chi f_0(H_1)E_1'(\lambda, h)\chi)$ plus p smooth terms with asymptotic expansion in h (Proposition 2.5).

We split (2.11) into two terms that we are going to study independently

$$\mathrm{Tr}\left(\left(\frac{\partial E_\varepsilon}{\partial \lambda} - \frac{\partial E_0}{\partial \lambda} \right)(1-\chi^2)f_0(H_0) \right), \tag{2.12}$$

$$\mathrm{Tr}\left(\frac{\partial E_\varepsilon}{\partial \lambda}(1-\chi^2)(f_0(H_\varepsilon)-f_0(H_0))\right). \tag{2.13}$$

Recall that these traces makes sense as distributions which means that they must be tested against a Schwartz function of λ. They are well defined since $q \in \mathscr{B}$.

2.4.1. Analysis of (2.12)

We use a microlocal partition of unit

$$(1-\chi)^2 f_0(H_0) = \chi^+(x,hD) + \chi^-(x,hD) \tag{2.14}$$

where $\mathrm{supp}\,\chi^\pm \subset \Gamma^\pm(R,J,\sigma^\pm)$ (see the next section for the notations). Then after a semiclassical Fourier transform and by a usual trick of Robert [33,34], we are left with the study of

$$\mathrm{Tr}(U_\varepsilon(t)\chi_\pm(x,hD) - U_0(t)\chi_\pm(x,hD)), \quad t \geqslant 0. \tag{2.15}$$

From now on we only consider the case + (the other one is analogous). By (3.2) we can write (2.15) as the sum

$$\begin{aligned}&\mathrm{Tr}(A_N^+(h,\varepsilon)U_0(t)B_N^+(h,\varepsilon)^\star - A_N^+(h,0)U_0(t)B_N^+(h,0)^\star)\\ &\quad + h^N\,\mathrm{Tr}(R_N^+(h,\varepsilon,t)), \quad t\geqslant 0.\end{aligned} \tag{2.16}$$

From Theorem 3.5 and Lemma 2.4 we can see easily that for any $j \geqslant 1$, the inverse Fourier transform of $\partial_\varepsilon^j\,\mathrm{Tr}(R_N^+(h,\varepsilon,t))1_{t\geqslant 0}$ is $\mathcal{O}(h^{k(N)})$ in $C^{k(N)}(J)$ with $k(N)\uparrow\infty$. If $j=0$, this term can be written $\mathrm{Tr}(\Upsilon_{N,1}^+ R(\lambda+i0,h))$ where $\Upsilon_{N,1}^+$ has the same properties as Υ_N^+ in Theorem 2.1. This remainder is the same as for the usual SSF in [34].

The first term of (2.16) can be written

$$\mathrm{Tr}((B_N^+(h,\varepsilon)^\star A_N^+(h,\varepsilon) - B_N^+(h,0)^\star A_N^+(h,0))U_0(t))$$

using Robert's cyclicity trick (see [7,34]). Then (3.14) shows that this term is still $\mathcal{O}(h^{k(N)})$ in $C^{k(N)}(J)$.

2.4.2. Analysis of (2.13)

First of all we remark that, up to terms which can be studied as above, we can insert an operator $f_1(H_0)$ between $(1-\chi^2)$ and $f_0(H_\varepsilon)-f_0(H_0)$, provided $f_1 \in C_0^\infty(I)$ and $f_1 \equiv 1$ near $\mathrm{supp}\,f_0$. This is an easy consequence of Lemma 2.2. Thus we get $(1-\chi^2)f_1(H_0)$ which we split into two terms $\tilde{\chi}^\pm(x,hD)$, using the same kind of partition of unit as in (2.14). Then semiclassical Fourier transform of (2.13) is the sum of

$$\mathrm{Tr}(U_0(t)\tilde{B}_N^\pm(h,\varepsilon)^\star(f_0(H_\varepsilon)-f_0(H_0))\tilde{A}_N^\pm(h,\varepsilon)), \quad \pm t\geqslant 0 \tag{2.17}$$

and some remainder terms which can be analyzed as (2.12). Now we use the pseudodifferential expansion of $f(H_\varepsilon) \sim \sum_k h^k c_k(x, hD, \varepsilon)$, with $\partial^\gamma_{x,\xi}\partial^j_\varepsilon c_k(x,\xi,\varepsilon) = \mathcal{O}(\langle x \rangle^{-j\rho})$ (see [7] for the details). Up to other remainders similar to those of (2.12) we obtain an expansion in powers of h with terms $\mathrm{Tr}(U_0(t)\mathscr{A}^\pm_{N,k}(h,\varepsilon))$ where

$$\mathscr{A}^\pm_{N,k}(h,\varepsilon) = \tilde{B}^\pm_N(h,\varepsilon)^\star c_k(x, hD, \varepsilon)\tilde{A}^\pm_N(h,\varepsilon).$$

Then we can calculate the Taylor expansion using (3.12), (3.13) and Proposition 3.4. This shows that

$$\mathscr{A}^\pm_{N,k}(h,\varepsilon) - \sum_{j=0}^{p-1} \frac{1}{j!}\partial^j_\varepsilon \mathscr{A}^\pm_{N,k}(h,\varepsilon)_{|\varepsilon=0} = \tilde{a}_{N,k}(x, hD, h),$$

where $\tilde{a}_{N,k} \in S_1(-\infty, p\rho)$ (compactly supported in $\omega^{-1}(I)$ w.r.t. ξ) with a full expansion in powers of h. By inverse Fourier transform and the stationary phase theorem, we obtain an expansion in $C^\infty(J)$ using the same method as in [34].

2.5. Conclusion

We have proved the theorem provided $q \in \mathscr{B}$. All the terms that we obtain in the final expansion depend continuously on $q \in S_1(\omega, -\rho)$. Note moreover that all the symbols and phases of Isozaki–Kitada's parametrix depend continuously on $q \in \mathscr{B}$ (see the next section). Therefore an easy density argument yields the result in the general case and completes the proof of Theorem 2.1.

3. Isozaki–Kitada's parametrix with parameters

3.1. A review of the construction

Let I be an open interval in $(0, \infty)$, noncritical for ω, i.e. $\omega(\xi) \in I \Rightarrow \nabla\omega(\xi) \neq 0$. For any $R > 0$, any J open interval such that $J \Subset I$ and $-1 < \sigma < 1$, one defines the outgoing area $\Gamma^+(R, J, \sigma)$ (resp. incoming Γ^-) by

$$\Gamma^\pm(R, J, \sigma) = \{(x,\xi) \in \mathbb{R}^{2d} \mid |x| > R,\ \omega(\xi) \in J,\ \pm\cos(x, v(\xi)) > \sigma\}, \qquad (3.1)$$

where $v(\xi) = \nabla\omega(\xi)$ and $\cos(x,y) = \langle \hat{x}, \hat{y} \rangle$ with $\hat{z} = z/|z|$.

The Isozaki–Kitada parametrix (see [24]) is a microlocal approximation of $U_\varepsilon(t) = \exp(-itH_\varepsilon/h)$ of the following form

$$U_\varepsilon(t)\chi_\pm(x, hD) = A^\pm_N(h,\varepsilon)U_0(t)B^\pm_N(h,\varepsilon)^\star + h^N R^\pm_N(h,\varepsilon,t), \quad \pm t \geqslant 0. \qquad (3.2)$$

Here χ_+ (resp. χ_-) is supported in the outgoing (resp. incoming) area $\Gamma^+(R, J, \sigma_+)$ (resp. $\Gamma^-(R, J, \sigma^-)$), $R^\pm_N(h,\varepsilon,t)$ are remainders that we will describe further on, and

$A_N^{\pm}(h,\varepsilon), B_N^{\pm}(h,\varepsilon)$ are Fourier Integral operators (FIO in the sequel) with phase functions *independent* of t. Namely we have $A_N^{\pm}(h,\varepsilon) = \sum_{k \leqslant N} h^k J_{\varphi^{\pm}}(a_k^{\pm})$ where

$$J_{\varphi^{\pm}}(a^{\pm})u(x) = (2\pi h)^{-d} \int \int e^{i(\varphi^{\pm}(x,\xi,\varepsilon,h) - \langle y,\xi \rangle)/h} a^{\pm}(x,\xi,\varepsilon,h)u(y)\, dy\, d\xi, \quad u \in \mathscr{S}(\mathbb{R}^d),$$

and the same with b instead of a for B.

Usually, the phase functions $\varphi^{\pm}$ and the symbols $a_k^{\pm}, b_k^{\pm}$ are independent of h; here we allow a h-dependence, for some minor technical reasons, and the reader may forget it in the beginning. Moreover, for $H_\varepsilon = -h^2 \Delta + \varepsilon V$, the phases does not depend on h.

The main point of this section is to study the behavior of Isozaki–Kitada's parametrix with respect to ε, especially when one differentiates with respect to this parameter. The usual estimates on $\varphi^{\pm}$ are $|\partial_x^\alpha(\varphi^{\pm}(x,\xi) - \langle x,\xi \rangle)| = \mathcal{O}(\langle x \rangle^{1-\rho-|\alpha|})$. In this section, our goal is to show that $\varphi^{\pm}$ depends smoothly on ε and that we get a power $\langle x \rangle^{-\rho}$ at each differentiation with respect to ε. We also need to control the constructions with respect to the principal symbol of the perturbation, $q \in \mathscr{B}$ (see the previous section).

We only treat the outgoing case, since the incoming is similar and we drop the subscript $+$ on the phases and the symbols.

Let $\sigma_0 \in (-1,1)$, and J_0 an open interval such that $J_0 \Subset I$. We note

$$H_\varepsilon^{cl}(x,\xi,\varepsilon,h) = \omega(\xi) + \varepsilon q(x,\xi,h).$$

The next proposition solves the Hamilton–Jacobi equation (3.3) and gives the properties of the phase.

Proposition 3.1. *There exists $R_0 > 0$ such that for all $q \in \mathscr{B}$, one can find a function $\varphi \in C^\infty(\mathbb{R}^{2d} \times [0,1] \times [0,h_0], \mathbb{R})$ satisfying*

$$H_\varepsilon^{cl}(x, \partial_x \varphi(x,\xi,\varepsilon,h), h) = \omega(\xi), \tag{3.3}$$

for all $(x,\xi) \in \Gamma^+(R_0, J_0, \sigma_0)$, $\varepsilon \in [0,1]$ and $h \in [0,h_0]$. φ satisfies the additional conditions:

$$\varphi(x,\xi,0,h) = \langle x,\xi \rangle, \tag{3.4}$$

$$|\nabla_x^t \nabla_\xi \varphi(x,\xi,\varepsilon,h) - 1_d| \leqslant 1/2, \tag{3.5}$$

$$|\partial_x^\alpha \partial_\xi^\beta \partial_\varepsilon^n \partial_h^l (\varphi(x,\xi,\varepsilon,h) - \langle x,\xi \rangle)| \leqslant C(\alpha,\beta,n,l) \langle x \rangle^{1-|\alpha|-\rho n_+}, \tag{3.6}$$

for all $(x, \xi, \varepsilon, h) \in \mathbb{R}^{2d} \times [0, 1] \times [0, h_0]$ *and* α, β, n, l, *with the notation* $n_+ = \max(1, n)$. *Moreover we can choose the function* φ *depending continuously on* q, *that is*

$$|\partial_x^\alpha \partial_\xi^\beta \partial_\varepsilon^n \partial_h^l (\varphi_1(x, \xi, \varepsilon, h) - \varphi_2(x, \xi, \varepsilon, h))| \leqslant C'(\alpha, \beta, n, l) \mathcal{N}(q^1 - q^2) \langle x \rangle^{1-|\alpha|-\rho n_+} \quad (3.7)$$

for some semi-norm $\mathcal{N}$ *on* $S_1(\omega, -\rho)$, *when* φ_j *is associated to* q^j, $j = 1, 2$.

This proposition will be proved in the next subsection. In the present subsection, we are only interested in the construction of the parametrix and we quote the results that we are going to use.

Once the phase function is defined, we have to consider $X = X(t, x, \xi, \varepsilon, h)$, the solution of the following differential equation:

$$\begin{cases} \dot{X} = \mathscr{V}(X, \xi, \varepsilon, h), \\ X_{|_{t=0}} = x, \end{cases}$$

where $\dot{X} = \partial_t X$ and $\mathscr{V}$ is the vector field defined by

$$\mathscr{V}(x, \xi, \varepsilon, h) \nabla_x = (\nabla_\xi H_\varepsilon^{cl})(x, \partial_x \varphi(x, \xi, \varepsilon, h), h) \nabla_x.$$

Let J_1 be an open interval such that $J_1 \Subset J_0$, and let σ_1 be such that $-1 < \sigma_1 < \sigma_0$.

Proposition 3.2. *There exists* $R_1 > 0$ *and* $e_1 > 0$ *such that, for all* $(x, \xi) \in \Gamma^+(R_1, J_1, \sigma_1)$, $(\varepsilon, h) \in [0, 1] \times [0, h_0]$, *and* $q \in \mathscr{B}$, *the solution* $X(t, x, \xi, \varepsilon, h)$ *is defined for* $t \geqslant 0$, *and satisfies*

$$|X(t, x, \xi, \varepsilon, h)| \geqslant e_1(t + |x|), \quad (3.8)$$

$$(X(t, x, \xi, \varepsilon, h), \xi) \in \Gamma^+(R_0, J_0, \sigma_0). \quad (3.9)$$

Moreover, for all α, β, n, l *there exists* $C > 0$ *such that*

$$|\partial_x^\alpha \partial_\xi^\beta \partial_\varepsilon^n \partial_h^l (X(t, x, \xi, \varepsilon, h) - x - tv(\xi))| \leqslant C \langle t \rangle \langle x \rangle^{-|\alpha|-\rho n_+}, \quad (3.10)$$

for all $(x, \xi) \in \Gamma^+(R_1, J_1, \sigma_1)$, $(\varepsilon, h) \in [0, 1] \times [0, h_0]$, $t \geqslant 0$ *and* $q \in \mathscr{B}$.

More generally, if the subscripts j refer to functions associated to q^j, $j = 1, 2$, *then, for all* α, β, n, l, *there exists a constant C and a semi-norm* $\mathcal{N}$ *in* $S_1(\omega, -\rho)$ *such that*

$$|\partial_x^\alpha \partial_\xi^\beta \partial_\varepsilon^n \partial_h^l (X_1 - X_2)(t, x, \xi, \varepsilon, h)| \leqslant C \mathcal{N}(q^1 - q^2) \langle t \rangle \langle x \rangle^{-|\alpha|-\rho n_+} \quad (3.11)$$

with respect to the same variables as in (3.10).

Proof. We use Proposition A.5 to prove (3.8) and (3.9), and Proposition A.1, applied to $y = X - x - tv(\xi)$ and $\eta = 0$ to prove (3.10). The details are essentially the same as for the previous proposition (proved in the next subsection) and we do not repeat the details. $\square$

Now we are in position to define the symbols $a_0, a_1, \dots, a_N$ in $\Gamma^+(R_1, J_1, \sigma)$. For $j = 0$

$$a_0(x, \xi, \varepsilon, h) = \exp\left(\int_0^{+\infty} p_0(X(t, x, \xi, \varepsilon, h), \xi, \varepsilon, h)\, dt\right),$$

and for $j \geqslant 1$ we have

$$a_j(x, \xi, \varepsilon, h) = \int_0^{+\infty} p_j(X(t, x, \xi, \varepsilon, h), \xi, \varepsilon, h) \exp\left(\int_0^t p_0(X(s, x, \xi, \varepsilon, h), \xi, \varepsilon, h)\, ds\right) dt.$$

The symbols $p_0, p_1, \dots$ are defined inductively by

$$p_0(x, \xi, \varepsilon, h) = \frac{1}{2}\operatorname{tr}((\partial^2_{\xi,\xi} H^{cl}_\varepsilon)(x, \partial_x \varphi(x, \xi, \varepsilon, h), h)(\partial^2_{x,x}\varphi)(x, \xi, \varepsilon, h)),$$

$$p_j = \sum_{j_0 + j_1 + j_2 = j+1,\ j_1 < j} (\tilde{H}^{cl}_{j_0} \# a_{j_1})_{j_2}, \quad j \geqslant 1,$$

where $\tilde{H}^{cl}_{j_0} = q_{j_0}$ if $j_0 > d$, $\tilde{H}^{cl}_0 = H^{cl}_\varepsilon$, $\tilde{H}^{cl}_{j_0} = 0$ for $1 \leqslant j_0 < d$, and $(H\#a)_j$ is the jth term of the symbol of $H(x, hD)J_\varphi(a)$. It is not hard to see that we have the following estimates for $(x, \xi) \in \Gamma^+(R_1, J_1, \sigma)$, $\varepsilon \in [0, 1]$, $h \in [0, h_0]$ and $q \in \mathscr{B}$:

$$|\partial_x^\alpha \partial_\xi^\beta \partial_\varepsilon^n \partial_h^l (a_0(x, \xi, \varepsilon, h) - 1)| \leqslant C\langle x\rangle^{-|\alpha| - \rho n}, \tag{3.12}$$

$$|\partial_x^\alpha \partial_\xi^\beta \partial_\varepsilon^n \partial_h^l a_k(x, \xi, \varepsilon, h)| \leqslant C\langle x\rangle^{-k-|\alpha| - \rho n}, \quad k \geqslant 1. \tag{3.13}$$

Note that all these symbols depend continuously on $q \in \mathscr{B}$ thanks to (3.7) and (3.11). The estimate (3.12) shows that, by choosing R_1 large enough, one can assume that $a_0 \geqslant 1/2$ in the outgoing area that we consider ($A_N(\varepsilon, h)$ is then said to be elliptic in this area). Then, after multiplication by a cutoff function in $S_1(-\infty, 0)$ supported in $\Gamma^+(R_1, J_1, \sigma)$ and which is $\equiv 1$ near $\Gamma^+(R_2, J_2, \sigma_2)$, for arbitrary $R_2 > R_1$, $J_2 \Subset J_1$ and $-1 < \sigma_2 < \sigma_1$, the symbols $a_0, \dots, a_N$ are globally defined.

Then we are able to construct the symbols $b_0, b_1, \dots, b_N$; they are defined inductively by requiring that

$$A_N(h, \varepsilon) B_N(h, \varepsilon)^\star = \chi_+(x, hD) + h^{N+1} r_N(x, hD, h, \varepsilon) \tag{3.14}$$

with $r_N \in S_1(-\infty, -N)$ (see (3.19)). This is, of course, a well-known consequence of the Egorov's theorem. However, since our result is based on a careful control of the constructions with respect to ε, we give a quite detailed proof. The kernel of

$A_N(h,\varepsilon)B_N(h,\varepsilon)^{\star}$ is the sum, for $k_1, k_2 \leqslant N$ of

$$\frac{h^{k_1+k_2}}{(2\pi h)^d} \int e^{i(\varphi(x,\xi,\varepsilon,h)-\varphi(y,\xi,\varepsilon,h))/h} a_{k_1}(x,\xi,\varepsilon,h)\overline{b_{k_2}(y,\xi,\varepsilon,h)}\, d\xi. \tag{3.15}$$

Then, by Taylor's formula, we write

$$\varphi(x,\xi,\varepsilon,h) - \varphi(y,\xi,\varepsilon,h) = \langle x-y, \tilde{\eta}(x,y,\xi,\varepsilon,h)\rangle$$

where $\tilde{\eta}(x,y,\xi,\varepsilon,h) = \int_0^1 \nabla_x \varphi(y+u(x-y),\xi,\varepsilon,h)\, du$ satisfies properties which we sum up in the following lemma.

Lemma 3.3. (i) *The map $\xi \mapsto \tilde{\eta}(x,y,\xi,\varepsilon,h)$ is a global diffeomorphism from $\mathbb{R}^d$ onto itself.*

(ii) *If we call $\tilde{\xi}(x,y,.,\varepsilon,h)$ its inverse, we have the following estimates:*

$$C_0^{-1}\langle \eta \rangle \leqslant \langle \tilde{\xi}(x,y,\eta,\varepsilon,h)\rangle \leqslant C_0 \langle \eta \rangle, \quad \textit{for some } C_0 > 0, \tag{3.16}$$

$$\begin{aligned} &|\partial_x^{\alpha}\partial_y^{\alpha'}\partial_\eta^{\beta}\partial_\varepsilon^{n}\partial_h^{n}(\tilde{\xi}(x,y,\eta,\varepsilon,h)-\eta)| \\ &\quad \leqslant C_{\alpha,\alpha',\beta,n,l}\langle x\rangle^{-m}\langle y\rangle^{-m'-\rho n_+}\langle x-y\rangle^{m+m'+\rho n_+} \end{aligned} \tag{3.17}$$

for all $x,y,\eta \in \mathbb{R}^d$, $\varepsilon \in [0,1]$, $h \in [0,h_0]$, and m,m' such that $m \leqslant |\alpha|$, $m' \leqslant |\alpha'|$. Moreover all these estimates are uniform with respect to $q \in \mathscr{B}$ and we have continuity estimates for $\tilde{\xi}$ analogous to (3.7).

Proof. The fact that we get a global diffeomorphism is a simple consequence (3.5) and to a global inversion theorem of [36]. Then the estimates follows from Proposition 3.1. □

After the change of variable $\xi = \tilde{\xi}(x,y,\eta,\varepsilon,h)$, (3.15) can be written as the sum, for $k_1, k_2 \leqslant N$, of the kernels of $(2\pi h)^{-d} h^{k_1+k_2} c_{k_1,k_2}(x,hD,\varepsilon,h)$, where

$$c_{k_1,k_2}(x,\eta,\varepsilon,h) \sim \sum_{\alpha} h^{|\alpha|}\frac{1}{\alpha!}\partial_y^{\alpha} D_\eta^{\alpha}\left(a_{k_1}(x,\tilde{\xi},\varepsilon,h)\overline{b_{k_2}(y,\tilde{\xi},\varepsilon,h)}\left|\frac{\partial \tilde{\xi}}{\partial \eta}\right|\right)_{|_{y=x}} \tag{3.18}$$

in which one must keep in mind that $\tilde{\xi} = \tilde{\xi}(x,y,\eta,\varepsilon,h)$. This allows us to compute explicitly $b_0,\dots,b_N$. This yields

$$\overline{b_0(x,\xi,\varepsilon,h)} = \chi_+(x,\tilde{\eta}(x,x,\xi,\varepsilon,h))\left(a_0(x,\xi,\varepsilon,h)\left|\frac{\partial \tilde{\xi}}{\partial \eta}(x,x,\tilde{\eta}(x,x,\xi,\varepsilon,h),\varepsilon,h)\right|\right)^{-1}$$

which is well defined since the term in $(\cdots)^{-1}$ cannot vanish on the support of $\chi_+(.,\tilde{\eta}(.,.,.,\varepsilon,h))$. The other terms are obtained by induction. We do not go any further into details and summarize the results on the symbols $b_0,\cdots,b_N$ in the following proposition.

Let $J_3 \Subset J_2$ and $-1<\sigma_3<\sigma_2$.

Proposition 3.4. *If χ_+ is supported in $\Gamma^+(R_3,J_3,\sigma_3)$* with *$R_3$ large enough, one can find symbols $b_0,\dots,b_N$ supported in $\Gamma^+(R_2,J_2,\sigma_2)$, depending smoothly on ε,h such that*:

$$|\partial_x^\alpha\partial_\xi^\beta\partial_\varepsilon^n\partial_h^l b_k(x,\xi,\varepsilon,h)|\leqslant C\langle x\rangle^{-k-|\alpha|-\rho n},\qquad 0\leqslant k\leqslant N,$$

and b_k depends continuously, in $S_1(-\infty,-k)$, on $q\in\mathscr{B}$. Moreover (3.14) *holds with $r_N(.,.,\varepsilon,h)$ in a bounded subset of $S_1(-\infty,-N)$, depending continuously on q and such that*

$$|\partial_x^\alpha\partial_\xi^\beta\partial_\varepsilon^n r_N(x,\xi,\varepsilon,h)|\leqslant C\langle x\rangle^{-N-|\alpha|-\rho n}. \tag{3.19}$$

All these results lead to the following theorem.

Theorem 3.5. *For all $J\Subset I$, and $-1<\sigma_+<1$, there exists $R>0$, such that, if* $\operatorname{supp}\chi_+\subset\Gamma^+(R,J,\sigma_+)$, *then $U_\varepsilon(t)\chi_+(x,hD)-A_N(\varepsilon,h)U_0(t)B_N(\varepsilon,h)^\star$ is equal to*

$$h^N\int_0^t U_\varepsilon(t-s)K_N(h,\varepsilon,s)\,ds+h^{N+1}U_\varepsilon(t)r_N(x,hD,\varepsilon,h), \tag{3.20}$$

for all $t\geqslant 0$, $h\in(0,h_0]$, $\varepsilon\in[0,1]$ and *$q\in\mathscr{B}$. $r_N(.,.,\varepsilon,h)$ is a symbol in $S_1(-\infty,-N)$ which depends smoothly on ε, and is bounded with respect to h. The kernel $\mathscr{K}_N(x,y,h,\varepsilon,s)$ of $K_N(h,\varepsilon,s)$ satisfies the following estimates*:

$$\partial_x^\alpha\partial_y^\beta\partial_\varepsilon^n K_N(x,y,h,\varepsilon,s)=\mathcal{O}(\langle x\rangle^{-[N/4]-|\alpha|}\langle y\rangle^{-[N/4]-|\beta|}\langle s\rangle^{-[N/4]}),$$

for all $s\geqslant 0$, $x,y\in\mathbb{R}^d$, $h\in(0,h_0]$, $\varepsilon\in[0,1]$ and uniformly with respect to $q\in\mathscr{B}$.

3.2. Proof of Proposition 3.1

Following [16,24,34], we first recall the formula which defines φ in some suitable outgoing area

$$\varphi(x,\xi,\varepsilon,h)=\langle x,\xi\rangle+\int_0^{+\infty}\frac{\partial F}{\partial t}(t,x,\xi,\varepsilon,h)\,dt, \tag{3.21}$$

where we have, for some $R>0$ large enough (whose existence will be proved below)

$$\frac{\partial F}{\partial t}(t,x,\xi,\varepsilon,h)=H_{\varepsilon}^{cl}(\tilde{z}(t,x,\xi,\varepsilon,h),\xi,h)-H_{\varepsilon}^{cl}(\tilde{z}(t,Rv(\xi),\xi,\varepsilon,h),\xi,h), \tag{3.22}$$

$$\tilde{z}(t,x,\xi,\varepsilon,h)=\bar{x}(t,x,\bar{\eta}(t,x,\xi,\varepsilon,h),\varepsilon,h) \tag{3.23}$$

with $\bar{\eta}(t,x,.,\varepsilon,h)$ the inverse of the diffeomorphism

$$\Gamma_x^+(R',J',\sigma')\ni\eta\mapsto\bar{\xi}(t,x,\eta,\varepsilon,h) \tag{3.24}$$

which is well defined on

$$\Gamma_x^+(R',J',\sigma')=\{\eta\in\mathbb{R}^d|(x,\eta)\in\Gamma^+(R',J',\sigma')\} \tag{3.25}$$

for a suitable outgoing area $\Gamma^+(R',J',\sigma')$ (see Lemma 3.7 below). Everywhere, $(\bar{x}(t,.),\bar{\xi}(t,.))$ denotes the classical flow, that is the solution to

$$\begin{cases}\partial_t\bar{x}=\partial_\xi H_\varepsilon^{cl}(\bar{x},\bar{\xi},h), & \bar{x}(0,x,\xi,\varepsilon,h)=x,\\ \partial_t\bar{\xi}=-\partial_x H_\varepsilon^{cl}(\bar{x},\bar{\xi},h), & \bar{\xi}(0,x,\xi,\varepsilon,h)=\xi.\end{cases}$$

The proof only consists in a technical review of proofs of [16,34] showing that all the objects can be constructed and estimated uniformly with respect to the parameters ε,h and q.

Let $J\Subset I$, and $\sigma\in(-1,1)$. The first step is to estimate the classical flow; this is the goal of the next lemma where $(\bar{x}_1,\bar{\xi}_1)$ and $(\bar{x}_2,\bar{\xi}_2)$ denote the respective Hamiltonian flows of $\omega+\varepsilon q_1$ and $\omega+\varepsilon q_2$ with $q_1,q_2\in\mathscr{B}$.

Lemma 3.6. *There exists $R>0$ such that, for all γ defined by $\partial^\gamma=\partial_x^\alpha\partial_\xi^\beta\partial_\varepsilon^n\partial_h^l$, one can find $C>0$ and $\mathcal{N}$* semi-norm *in $\mathscr{S}_1(\omega,-\rho)$ such that*

$$|\partial^\gamma(\bar{x}_1-\bar{x}_2)(t,x,\xi,\varepsilon,h)|\leqslant C\mathcal{N}(q_1-q_2)\langle t\rangle\langle x\rangle^{-|\alpha|-\rho n_+},$$

$$|\partial^\gamma(\bar{\xi}_1-\bar{\xi}_2)(t,x,\xi,\varepsilon,h)|\leqslant C\mathcal{N}(q_1-q_2)\langle x\rangle^{-|\alpha|-\rho n_+},$$

for all $(x,\xi)\in\Gamma^+(R,J,\sigma)$, $(\varepsilon,h)\in[0,1]\times[0,h_0]$, $t\geqslant0$ and $q_1,q_2\in\mathscr{B}$.

Proof. By reviewing the proof of Lemma 2.2 of [16] (see also Proposition A.5 of this paper), one get easily the existence of $R>0$ and $c>0$ such that

$$|\bar{x}(t,x,\xi,\varepsilon,h)-x-tv(\xi)|\leqslant c\langle t\rangle\langle x\rangle^{-\rho}, \tag{3.26}$$

$$|\bar{\xi}(t,x,\xi,\varepsilon,h)-\xi|\leqslant c\langle x\rangle^{-\rho}, \tag{3.27}$$

$$|\bar{x}(t,x,\xi,\varepsilon,h)|\geqslant c^{-1}(|x|+t) \tag{3.28}$$

for all $t \geqslant 0$, $q \in \mathscr{B}$, $\varepsilon \in [0,1]$, $h \in [0, h_0]$ and $(x, \xi) \in \Gamma^+(R, J, \sigma)$. This shows that our lemma is true if $q_2 \equiv 0$. More precisely, we see that $y = \partial_\varepsilon \bar{x}$ and $\eta = \partial_\varepsilon \bar{\xi}$ are solutions of system (A.1) with $y_0 = \partial_\xi q(\bar{x}, \bar{\xi})$, $\eta_0 = -\partial_x q(\bar{x}, \bar{\xi})$ and

$$A = \varepsilon \partial^2_{x,\xi} q(\bar{x}, \bar{\xi}), \quad B = \partial^2_{\xi,\xi}\omega(\bar{\xi}) + \varepsilon \partial^2_{\xi,\xi} q(\bar{x}, \bar{\xi}),$$

$$C = -\varepsilon \partial^2_{x,x} q(\bar{x}, \bar{\xi}), \quad D = -\varepsilon \partial^2_{x,\xi} q(\bar{x}, \bar{\xi}),$$

which satisfies (A.2), (A.3) since $q \in \mathscr{B}$ and thanks to the estimates (3.26), (3.27) and (3.28). More generally, by induction on $|\gamma| \geqslant 0$, we get the estimates on $\partial^\gamma \partial_\varepsilon \bar{x}$, $\partial^\gamma \partial_\varepsilon \bar{\xi}$ by means of Proposition A.1. This proves the result if $q_2 \equiv 0$. Then we get the general result by considering system (A.1) with $y = \bar{x}_2 - \bar{x}_1$, $\eta = \bar{\xi}_2 - \bar{\xi}_1$ and

$$\begin{aligned}
A &= \int_0^1 \varepsilon \partial^2_{x,\xi} q_2(\bar{x}_1 + u(\bar{x}_2 - \bar{x}_1), \bar{\xi}_1)\, du,\\
B &= \int_0^1 \partial^2_{\xi,\xi}\omega(\bar{\xi}_1 + u(\bar{\xi}_2 - \bar{\xi}_1)) + \varepsilon \partial^2_{\xi,\xi} q_2(\bar{x}_2, \bar{\xi}_1 + u(\bar{\xi}_2 - \bar{\xi}_1))\, du,\\
C &= -\int_0^1 \varepsilon \partial^2_{x,x} q_2(\bar{x}_1 + u(\bar{x}_2 - \bar{x}_1), \bar{\xi}_2)\, du,\\
D &= -\int_0^1 \varepsilon \partial^2_{x,\xi} q_2(\bar{x}_1, \bar{\xi}_1 + u(\bar{\xi}_2 - \bar{\xi}_1))\, du,
\end{aligned}$$

and with the second terms y_0 and η_0 given by

$$y_0 = \varepsilon(\partial_\xi q_2(\bar{x}_1, \bar{\xi}_1) - \partial_\xi q_1(\bar{x}_1, \bar{\xi}_1)),$$

$$\eta_0 = \varepsilon(\partial_x q_1(\bar{x}_1, \bar{\xi}_1) - \partial_x q_2(\bar{x}_1, \bar{\xi}_1)).$$

By increasing R if necessary (uniformly with respect to ε, h, q) so that

$$|\bar{x}_1 + u(\bar{x}_2 - \bar{x}_1)| \geqslant c'(|x| + t), \quad \forall u \in [0,1],$$

for some $c' > 0$, we see that A, B, C, D, y_0 and η_0 satisfy (A.2)–(A.4). This allows us to use again Proposition A.1 from which we get the result by applying ∂^γ to this new system, arguing by induction on $|\gamma| \geqslant 0$. $\square$

Lemma 3.7. *There exists $\tilde{R} \geqslant R$ such that, for all $|x| > \tilde{R}$, $t \geqslant 0$, $(\varepsilon, h) \in [0,1] \times [0, h_0]$ and $q \in \mathscr{B}$*

$$\Gamma^+_x(\tilde{R}, J, \sigma) \ni \eta \mapsto \bar{\xi}(t, x, \eta, \varepsilon, h)$$

is a diffeomorphism onto its range. We call $\bar{\eta}(t, x, ., \varepsilon, h)$ its inverse.

Proof. We use the same method as Gérard–Martinez in [16]: thanks to Lemma 3.6, $\partial_\eta \bar{\xi}$ is invertible when $|x| \geqslant \tilde{R} \gg 1$, so we have a local diffeomorphism. Thus it is sufficient to show that $\bar{\xi}$ is injective. This is a consequence of the fact that, if $\eta, \eta' \in \Gamma_x^+(\tilde{R}, J, \sigma)$

$$|\bar{\xi}(t,x,\eta,\varepsilon,h) - \bar{\xi}(t,x,\eta',\varepsilon,h)| \geqslant \tfrac{1}{2}|\eta - \eta'|, \quad \forall |x| \geqslant \tilde{R},\ t \geqslant 0,\ \varepsilon \in [0,1],\ h \in [0,h_0],\ q \in \mathscr{B}$$

if $\tilde{R}$ is large enough, by application of the Taylor formula to the second order, and using again the previous lemma. The proof is complete. $\square$

The following proposition controls the range of $(x,\eta) \mapsto (x, \bar{\xi}(t,x,\eta,\varepsilon,h))$ will be useful to define (3.23) (and hence (3.21) in an outgoing area independent of ε, h and q.

Proposition 3.8. *For all open interval J', with $J' \Subset J$ and all $-1 < \sigma' < \sigma$, there exists $R' > 0$ such that*

$$\Gamma^+(R', J', \sigma') \subset \Phi_{t,\varepsilon,h,q}(\Gamma^+(R, J, \sigma)) \tag{3.29}$$

for all $t \geqslant 0$, $(\varepsilon, h) \in [0,1] \times [0, h_0]$ and $q \in \mathscr{B}$, where $\Phi_{t,\varepsilon,h,q}$ denotes the map

$$\Gamma^+(R, J, \sigma) \ni (x, \eta) \mapsto (x, \bar{\xi}(t, x, \eta, \varepsilon, h))$$

which is a diffeomorphism onto its (open) range.

Moreover, we can choose R' such that, for all γ there exists $C \geqslant 0$ and $\mathscr{N}$ semi-norm in $\mathscr{S}_1(\omega, -\rho)$ such that, for all $(x,\xi) \in \Gamma^+(R', J', \sigma')$, $\varepsilon \in [0,1]$, $h \in [0,h_0]$ and $q_j \in \mathscr{B}$ we have

$$|\partial^\gamma(\bar{\eta}_1 - \bar{\eta}_2)(t,x,\xi,\varepsilon,h)| \leqslant C \mathscr{N}(q_1 - q_2) \langle x \rangle^{-|\alpha| - \rho n_+}, \tag{3.30}$$

where $\bar{\eta}_j$ is associated to q_j, $j = 1, 2$.

Proof. (3.29) is based on the following basic topological fact (see [33]): if $A, B \subset \mathbb{R}^n$ with A connected, and if $f : \bar{B} \to \mathbb{R}^n$ is an homeomorphism onto its range, then

$$A \cap f(B) \neq \emptyset \quad \text{and} \quad A \cap f(\partial B) = \emptyset \Rightarrow A \subset f(B).$$

To use this property, we write $\omega^{-1}(J') = \coprod_l \Omega_l$, where the Ω_l's are the open connected components of $\omega^{-1}(J')$. It is a simple exercise to show that, for all R', $\Gamma_l^+(R') = \{(x,\xi) \mid x| > R',\ \xi \in \Omega_l, \cos(x, v(\xi)) > -\sigma'\}$ is connected, thus it is sufficient to prove that:

$$\forall R' \geqslant R,\ \forall l,\ \forall t, \varepsilon, h, q, \qquad \Gamma_l^+(R') \cap \Phi_{t,\varepsilon,h,q}(\Gamma^+(R,J,\sigma)) \neq \emptyset, \tag{3.31}$$

$$\exists R' \geqslant R,\ \forall l,\ \forall t, \varepsilon, h, q, \quad \Gamma_l^+(R') \cap \Phi_{t,\varepsilon,h,q}(\partial \Gamma^+(R,J,\sigma)) = \emptyset. \tag{3.32}$$

(3.31) is a simple consequence of the fact that $\bar{\xi}(t,x,\xi,\varepsilon,h) \to \xi$ as $|x| \to \infty$, since this implies that $\Phi_{t,\varepsilon,h,q}(x,\xi) \in \Gamma_l^+(R')$ if $(x,\xi) \in \Gamma_l^+(R') (\subset \Gamma^+(R,J,\sigma))$ and $|x|$

large enough. In order to prove (3.32), we remark that, if $(x,\xi)\in\Gamma_l^+(R')\cap\Phi_{t,\varepsilon,h,q}(\partial\Gamma^+(R,J,\sigma))$, with $R'>R$ then there exists ξ' such that

$$\xi=\bar{\xi}(t,x,\xi',\varepsilon,h) \quad \text{and} \quad (\omega(\xi')\in\partial J \text{ or } \cos(x,v(\xi'))=-\sigma). \tag{3.33}$$

By Lemma 3.6, it is clear that, for any $\delta>0$, we have $|\xi-\xi'|<\delta$ if $R'\gg 1$ (uniformly with respect to t,ε,h,q). Since J is a neighborhood of $\bar{J}'$ and $\sigma\neq\sigma'$, (3.33) cannot hold true if R' is large enough. This completes the proof of the first point.

In order to prove estimate (3.30), we use the trivial relation

$$0=(\bar{\xi}_1(t,x,\bar{\eta}_1,\varepsilon,h)-\bar{\xi}_1(t,x,\bar{\eta}_2,\varepsilon,h))+(\bar{\xi}_1(t,x,\bar{\eta}_2,\varepsilon,h)-\bar{\xi}_2(t,x,\bar{\eta}_2,\varepsilon,h)). \tag{3.34}$$

Since, for any $\delta>0$, we can choose $R'>0$ big enough, such that $|\bar{\eta}_j-\xi|\leqslant\delta$, $(j=1,2)$ and $|\partial_\xi\bar{\xi}_1(t,x,u\bar{\eta}_1+(1-u)\bar{\eta}_2,\varepsilon,h)-1_d|\leqslant\delta$, for all $u\in[0,1]$, $(x,\xi)\in\Gamma^+(R',J',\sigma'))$, $\varepsilon\in[0,1]$, $h\in[0,h_0]$ and $q_j\in\mathscr{B}$, we get, by Taylor formula:

$$\int_0^1 \partial_\xi\,\bar{\xi}_1(t,x,u\bar{\eta}_1+(1-u)\bar{\eta}_2,\varepsilon,h)\,du(\bar{\eta}_2-\bar{\eta}_1)=\mathcal{O}(\mathcal{N}(q_1-q_2)\langle x\rangle^{-\rho})$$

which proves the result for $|\gamma|=0$, by Lemma 3.6. Then we get the general result by induction on $|\gamma|$ by differentiating the relation (3.34). We do not go further into details. $\square$

Proof of the proposition. First we mention the fact that φ, given by (3.21), satisfies the Hamilton–Jacobi equation (3.3). This is standard (see again [16,34]) and we only insist on the proof of the estimates.

Let J_0 and σ_0 be fixed and let us choose open intervals J,J' and real numbers σ,σ' such that $J_0\Subset J'\Subset J\Subset I$ and $-1<\sigma_0<\sigma'<\sigma<1$.

According to formula (3.21) and following [16], the estimate (3.6) will be a consequence of the fact that

$$|\partial^\gamma(\partial_x\tilde{z}(t,x,\xi,\varepsilon,h)-1_d)|\leqslant c_\gamma\langle x\rangle^{-|\alpha|-\rho n_+} \tag{3.35}$$

where $\partial^\gamma=\partial_x^\alpha\partial_\xi^\beta\partial_\varepsilon^n\partial_h^l$ and c_γ is a constant independent on $t\geqslant 0$, $(x,\xi)\in\Gamma^+(R',J',R')$, and ε,h,q. To prove this estimate, we study, when $0\leqslant s\leqslant t$,

$$\bar{x}(s,\bar{\eta})=\bar{x}(s,\bar{\eta}(t))=\bar{x}(s,x,\bar{\eta}(t,x,\xi,\varepsilon,h),\varepsilon,h),$$

$$\bar{\xi}(s,\bar{\eta})=\bar{\xi}(s,\bar{\eta}(t))=\bar{\xi}(s,x,\bar{\eta}(t,x,\xi,\varepsilon,h),\varepsilon,h)$$

which are solutions of the following equations:

$$\bar{x}(s,\bar{\eta}(t))=x+\int_0^s \partial_\xi\omega(\bar{\xi}(u,\bar{\eta}))+\varepsilon\partial_\xi q(\bar{x}(u,\bar{\eta}),\bar{\xi}(u,\bar{\eta}),h)\,du, \tag{3.36}$$

$$\bar{\xi}(s,\bar{\eta}(t)) = \xi + \int_s^t \varepsilon\partial_x q(\bar{x}(u,\bar{\eta}),\bar{\xi}(u,\bar{\eta}),h)\,du. \tag{3.37}$$

We remark that the following three estimates hold, for $(x,\xi)\in\Gamma^+(R',J',\sigma')$, $\varepsilon\in[0,1]$, $h\in[0,h_0]$, $q\in\mathscr{B}$ and $t\geqslant 0$, if we choose $R'>0$ large enough

$$|\bar{x}(s,\bar{\eta}(t))|\geqslant e_0(s+|x|), \tag{3.38}$$

$$|\partial^\gamma(\bar{x}(s,\bar{\eta}(t)) - x - sv(\xi))|\leqslant c_\gamma\langle s\rangle\langle x\rangle^{-\rho n_+-|\alpha|}, \tag{3.39}$$

$$|\partial^\gamma(\bar{\xi}(s,\bar{\eta}(t)) - \xi)|\leqslant c_\gamma\langle x\rangle^{-\rho n_+-|\alpha|}. \tag{3.40}$$

(3.38) is a consequence of (3.39) which, as (3.40), is a consequence of (3.30) and Lemma 3.6. Now we can prove (3.35) for $|\gamma|=0$ since, by applying ∂_x to (3.36) and (3.37), we find that (y,η) defined as

$$y = \partial_x(\bar{x}(s,\bar{\eta}(t)) - x), \tag{3.41}$$

$$\eta = \partial_x(\bar{\xi}(s,\bar{\eta}(t))), \tag{3.42}$$

is the solution of a system similar to (A.10) (notice that the fact that y and η are matrices and not vectors is irrelevant). Hence Proposition A.3 shows that, uniformly with respect to the parameters, we have

$$|y|\leqslant c|x|^{-\rho}, \qquad |\eta|\leqslant c(s+|x|)^{-1-\rho'}|x|^{\rho'-\rho}$$

(see Proposition A.3 for the definition of ρ'). Then we get, by induction on $|\gamma|\geqslant 0$, that

$$|\partial^\gamma y|\leqslant c_\gamma|x|^{-|\alpha|-\rho n_+}, \tag{3.43}$$

$$|\partial^\gamma \eta|\leqslant c_\gamma(s+|x|)^{-1-\rho'}|x|^{\rho'-|\alpha|-\rho n_+}, \tag{3.44}$$

again by application of Proposition A.3 and using the fact that, when $a\in S_1(\omega,-\mu)$ with $\mu\geqslant 0$,

$$\partial^\gamma(a(\bar{x}(s,\bar{\eta}(t)),\bar{\xi}(s,\bar{\eta}(t)))) = \mathcal{O}((s+|x|)^{-\mu}|x|^{-|\alpha|-\rho n}),$$

so the estimate (3.35) is proved. In the same way, (3.7) is essentially a consequence of the fact that

$$|\partial^\gamma(\partial_x\tilde{z}_1(t,x,\xi,\varepsilon,h) - \partial_x\tilde{z}_2(t,x,\xi,\varepsilon,h))|\leqslant c_\gamma\mathcal{N}(q_1-q_2)\langle x\rangle^{-|\alpha|-\rho n_+}$$

with $\tilde{z}_j$ associated to q_j. The proof of this new estimate is the same as above by mean of Proposition A.3 (with $C_1 = \mathcal{O}(\mathcal{N}(q_1 - q_2))$) to $y = y_1 - y_2$ and $\eta = \eta_1 - \eta_2$, if y_j and η_j are defined by (3.41) and (3.42) with $\bar{x}_j$ and $\bar{\eta}_j$, instead of $\bar{x}$ and $\bar{\eta}$. We can apply Proposition A.3, once we have chosen R' large enough so that

$$|\bar{x}_1 + u(\bar{x}_2 - \bar{x}_1)| \geqslant c'(|x| + t), \quad \forall u \in [0, 1]$$

with $c' > 0$, which allows to get the right estimates for the matrices involved in the system.

Finally, choosing $R_0 > R'$, and multiplying $\partial F/\partial t$ by a cutoff function in $S_1(-\infty, 0)$ supported in $\Gamma^+(R', J', \sigma')$ and identically 1 in a neighborhood of $\Gamma^+(R_0, J_0, \sigma_0)$, we get the result. $\square$

Appendix A. Estimates on solutions of some ODE

In this appendix, we are going to show estimates on $y = y(t, x)$ and $\eta = \eta(t, x)$ which are solutions of some ordinary differential equations, or integral equations.

The first proposition deals with the solutions of the following linear differential system (where $\dot{z} = \partial_t z$)

$$\begin{cases} \dot{y} = Ay + B\eta + y_0, \\ \dot{\eta} = Cy + D\eta + \eta_0, \end{cases} \tag{A.1}$$

where A, B, C, D are $d \times d$ matrices and y_0, η_0 are vectors of $\mathbb{R}^d$ all depending continuously on $(t, x) \in [0, \infty) \times \mathcal{E}(R)$, with $\mathcal{E}(R) \subset B(0, R)^c$, and satisfying the following estimates for some $C_0 \geqslant 0$, $C_1 \geqslant 0$ and $\mu \geqslant 0$

$$|A(t, x)| \leqslant C_0(t + |x|)^{-\rho-1}, \quad |B(t, x)| \leqslant C_0, \tag{A.2}$$

$$|C(t, x)| \leqslant C_0(t + |x|)^{-\rho-2}, \quad |D(t, x)| \leqslant C_0(t + |x|)^{-\rho-1}, \tag{A.3}$$

$$|y_0(t, x)| \leqslant C_1|x|^{-\mu-\rho}, \quad |\eta_0(t, x)| \leqslant C_1|x|^{-\mu}(t + |x|)^{-1-\rho} \tag{A.4}$$

for all $t \geqslant 0$ and $x \in \mathcal{E}(R)$ (the matrix norm, also denoted by $|.|$ is such that $|Ez| \leqslant |E|.|z|$).

Proposition A.1. *There exists $R \geqslant 1$ large enough, and $C_2 > 0$, both depending only on C_0 and ρ, such that for all $x \in \mathcal{E}(R)$, the solution (y, η) vanishing at $t = 0$ satisfies*

$$|y(t, x)| \leqslant C_2 C_1 \langle t \rangle |x|^{-\mu-\rho}, \tag{A.5}$$

$$|\eta(t,x)| \leqslant C_2 C_1 |x|^{-\mu-\rho}. \tag{A.6}$$

The proof is a consequence of the following lemma.

Lemma A.2. *Assume that* $C_1 = 1$. *Then for all* $0 < \varepsilon \leqslant 1$ *there exists* $R = R(C_0, \rho, \varepsilon) \geqslant 1$, *such that for all* $|x| \geqslant R$ *and* $t \geqslant 0$

$$|y(t,x)| \leqslant \varepsilon \langle t \rangle |x|^{\frac{\rho}{2}-\mu-\rho}, \quad |\eta(t,x)| \leqslant \varepsilon |x|^{\frac{\rho}{2}-\mu-\rho}.$$

Proof. Let us choose R such that

$$2e^{2C_0} R^{-\frac{\rho}{2}} \leqslant \varepsilon, \tag{A.7}$$

$$2C_0 \int_0^\infty (|x|+s)^{-\rho-1}\, ds + |x|^{\rho-\frac{\rho}{2}} \int_0^\infty (|x|+s)^{-1-\rho}\, ds \leqslant \frac{\varepsilon}{4}|x|^{-\frac{\rho}{4}}, \quad \forall\ x \in \mathscr{E}(R), \tag{A.8}$$

$$C_0 \int_0^\infty (R+s)^{-\rho-1}\, ds + C_0 R^{-\frac{\rho}{4}} + R^{-\frac{\rho}{2}} \leqslant \frac{\varepsilon}{2}. \tag{A.9}$$

We shall prove that for all $x \in \mathscr{E}(R)$, we have $I(x) = [0, \infty)$ where

$$I(x) = \{T \geqslant 0 \mid |y(t,x)| \leqslant \varepsilon \langle t \rangle |x|^{\frac{\rho}{2}-\mu-\rho} \quad \text{and } |\eta(t,x)| \leqslant \varepsilon |x|^{\frac{\rho}{2}-\mu-\rho}, \forall t \in [0,T]\}.$$

For $t \in [0,1]$ we have easily

$$|y(t,x)| + |\eta(t,x)| \leqslant 2C_0 \int_0^t |y(s,x)| + |\eta(s,x)|\, ds + 2|x|^{-\mu-\rho},$$

so by the Gronwall's lemma we get

$$|y(t,x)| + |\eta(t,x)| \leqslant 2|x|^{-\mu-\rho} e^{2C_0}, \quad \forall t \in [0,1]$$

and the right-hand side is lower than $\varepsilon |x|^{\frac{\rho}{2}-\mu-\rho}$ if (A.7) is satisfied, hence $I(x)$ is not empty. Moreover $I(x)$ is clearly an interval so we only have to prove that $T(x) := \sup I(x) = +\infty$. We argue by contradiction so we assume that $T(x) < \infty$. Then for all $t \in [0, T(x)]$, integration of the second equation of (A.1) between 0 and t yields

$$\begin{aligned} |y(t,x)| &\leqslant 2C_0 \int_0^t (|x|+s)^{-\rho-1} \varepsilon |x|^{\frac{\rho}{2}-\mu-\rho}\, ds + |x|^{-\mu} \int_0^t (s+|x|)^{-1-\rho}\, ds \\ &\leqslant \frac{\varepsilon}{4} |x|^{-\frac{\rho}{4}+\frac{\rho}{2}-\mu-\rho} \end{aligned}$$

if (A.8) is satisfied; then using this new estimate and by integrating the first equation of (A.1) between 0 and t we get

$$\begin{aligned}|y(t,x)| &\leqslant \langle t\rangle |x|^{\frac{\rho}{2}-\mu-\rho}\left(\varepsilon C_0 \int_0^t (s+|x|)^{-1-\rho}\, ds + \varepsilon C_0|x|^{-\frac{\rho}{4}} + |x|^{-\frac{\rho}{2}}\right)\\ &\leqslant \frac{\varepsilon}{2}\langle t\rangle |x|^{\frac{\rho}{2}-\mu-\rho}\end{aligned}$$

if (A.9) is satisfied. This shows that the estimates on y and η are valid on $[0, T']$ for some $T' > T(x)$ which is a contradiction. □

Proof of the proposition. If $C_1 = 0$, $y = \eta = 0$ and the result is trivial. If $C_1 > 0$, then $y/C_1, \eta/C_1$ is solution of the same system with y_0, η_0 replaced by $y_0/C_1, \eta_0/C_1$ which satisfies $|y_0/C_1| \leqslant |x|^{-\mu-\rho}$ and $|\eta_0/C_1| \leqslant |x|^{-\mu}(t+|x|)^{-1-\rho}$, so we may use Lemma A.2. We choose $\varepsilon = 1$, then for some $R \geqslant 1$ depending only on C_0 and ρ, we have

$$|y(t,x)/C_1| \leqslant \langle t\rangle |x|^{\frac{\rho}{2}-\mu-\rho}, \quad \text{and} \quad |\eta(t,x)/C_1| \leqslant |x|^{\frac{\rho}{2}-\mu-\rho}, \quad \forall t \geqslant 0, \ |x| \geqslant R.$$

Then by integrating the second line of (A.1) we get

$$\begin{aligned}|\eta(t,x)/C_1| \leqslant{}& C_0 \int_0^\infty (s+|x|)^{-2-\rho}(1+s)|x|^{\frac{\rho}{2}-\mu-\rho} ds\\ &+ C_0 \int_0^\infty (s+|x|)^{-1-\rho}|x|^{\frac{\rho}{2}-\mu-\rho}\, ds + |x|^{-\mu} \int_0^\infty (s+|x|)^{-1-\rho}\, ds\\ \leqslant{}& C_2|x|^{-\mu-\rho}\end{aligned}$$

with C_2 depending only on C_0 and ρ. In the same way we get the expected estimate on y/C_1 (by using the final estimate on η/C_1). The proof is complete. □

Our second proposition gives estimates on the solutions of the following system of integral equations

$$\begin{cases} y(s) = y_0(s) + \int_0^s A(u)y(u)\, du + \int_0^s B(u)\eta(u)du, \\ \eta(s) = \eta_0(s) + \int_s^t C(u)y(u)\, du + \int_s^t D(u)\eta(u)\, du, \end{cases} \tag{A.10}$$

where $0 \leqslant s \leqslant t$, and $y = y(s) = y(s,x)$ (as well as η, y_0, η_0, A, B, C and D). Here again A, B, C, D are $d \times d$ matrices satisfying (A.2), (A.3) and y_0, η_0 vectors satisfying (A.4). We assume moreover that the solution y, η of the system satisfies the following a priori estimates:

$$|y| \leqslant C_1 \langle s\rangle |x|^{-\rho-\mu-1}, \tag{A.11}$$

$$|\eta| \leqslant C_1 |x|^{-\rho-\mu-1}. \tag{A.12}$$

Precisely we have

Proposition A.3. *Assume that $\rho^{-1} \notin \mathbb{N}$. There exists C_2 which depends only on ρ and C_0, such that*

$$|y| \leqslant C_2 C_1 |x|^{-\rho-\mu}, \tag{A.13}$$

$$|\eta| \leqslant C_2 C_1 (s + |x|)^{-1-\rho'} |x|^{\rho'-\rho-\mu}, \tag{A.14}$$

where $\rho' > 0$ is defined by $1 + \rho' = (k+1)\rho$, if $k \geqslant 0$ is the lowest integer such that $(k+1)\rho > 1$.

This proposition gives a L^∞ estimate for y with respect to s, and a L^1 one for η, which is crucial in the proof of Proposition 3.1.

Proof. Putting the a priori estimates (A.11) and (A.12) in the second equation of the system (A.10) yields

$$\eta(s) = \mathcal{O}(C_1 (s + |x|)^{-\rho} |x|^{-\mu-1})$$

which means that $|\eta(s)| \leqslant c C_1 (s + |x|)^{-\rho} |x|^{-\mu-1}$ for some c depending only on ρ and C_0. Then using this new estimate together with (A.11) in the first equation of (A.10) we get

$$y(s) = \mathcal{O}(C_1 |x|^{-\rho-\mu}) + \mathcal{O}\left(C_1 \int_0^s (u + |x|)^{-\rho} |x|^{-\mu-1} \, du \right)$$

and there are two cases: either $\rho > 1$: then we get the result or $\rho < 1$, and we have the estimate

$$y(s) = \mathcal{O}(C_1 |x|^{-\rho-\mu}) + \mathcal{O}(C_1 (s + |x|)^{1-\rho} |x|^{-\mu-1})$$

so we are in position to apply the following lemma from which the proof of the proposition is a trivial consequence by a finite induction. □

Lemma A.4. *Assume that for some $n \geqslant 1$, we have*

$$\eta(s) = \mathcal{O}(C_1 (s + |x|)^{-\rho-1} |x|^{-\mu}) + \mathcal{O}(C_1 (s + |x|)^{-n\rho} |x|^{-1-\mu}), \tag{A.15}$$

$$y(s) = \mathcal{O}(C_1 |x|^{-\rho-\mu}) + \mathcal{O}(C_1 (s + |x|)^{1-n\rho} |x|^{-1-\mu}), \tag{A.16}$$

then we have

$$\eta(s) = \mathcal{O}(C_1 (s + |x|)^{-\rho-1} |x|^{-\mu}) + \mathcal{O}(C_1 (s + |x|)^{-(n+1)\rho} |x|^{-1-\mu}), \tag{A.17}$$

and

$$y(s) = \begin{cases} \mathcal{O}(C_1|x|^{-\rho-\mu}) + \mathcal{O}(C_1(s+|x|)^{1-(n+1)\rho}|x|^{-1-\mu}), & \textit{if } 1-(n+1)\rho>0 \\ \mathcal{O}(C_1|x|^{-\rho-\mu}), & \textit{if } 1-(n+1)\rho<0. \end{cases}$$

Proof. Putting the estimates (A.16) and (A.15) into the second equation of the system (A.10) leads directly to the estimate (A.17). Then by putting (A.17) and (A.16) into the first equation of the system, one proves that

$$y(s) = \mathcal{O}(C_1|x|^{-\rho-\mu}) + \mathcal{O}\left(C_1 \int_0^s (u+|x|)^{-(n+1)\rho}|x|^{-\mu-1}\, du\right)$$

from which we deduce the estimates on $y(s)$ in each case, and this completes the proof. $\square$

Our final result gives estimates on $y = y(t) = y(t,x,\xi)$, the solution of

$$\begin{cases} \dot{y} = v(\xi) + w(y,\xi), \\ y_{|_{t=0}} = x, \end{cases} \tag{A.18}$$

for $(x,\xi)\in\Gamma^+(R_0,J_0,\sigma_0)$. We assume that $w\in S_1(\omega,-\rho)$ (with values in $\mathbb{R}^d$), that is, for all α,β

$$|\partial_x^\alpha\partial_\xi^\beta w(x,\xi)| \leqslant C_{\alpha\beta}(1+\omega(\xi))\langle x\rangle^{-\rho-|\alpha|}.$$

Then we have

Proposition A.5. *There exists $\tilde{R}_0\geqslant R_0$ and c_0 (both depending only on J_0, σ_0 and a finite number of semi-norms of w in $S_1(\omega,-\rho)$) such that, for all $(x,\xi)\in\Gamma^+(\tilde{R}_0,J_0,\sigma_0)$, $y(t,x,\xi)$ is defined for all $t\geqslant 0$ and*

$$|y(t,x,\xi) - x - tv(\xi)| \leqslant c_0\langle t\rangle\langle x\rangle^{-\rho}, \quad \forall t\geqslant 0.$$

Proof. By differentiating the expression $\langle \dot{y}, y\rangle$ with respect to t, we get easily

$$\frac{\partial^2}{\partial t^2}|y|^2 \geqslant 2|\dot{y}|^2 - C\langle y\rangle^{-\rho} \tag{A.19}$$

where C depends only on J_0 and some semi-norms of w. Independently, let us remark that we can choose $\delta_0, e_0>0$ small enough and $\tilde{R}_0$ large enough (all depending only on J_0,σ_0 and some semi-norms of w) such that

$$(1-\delta_0)t^2|v(\xi)|^2 + 2t\langle x, v(\xi)+w(x,\xi)\rangle + |x|^2 \geqslant e_0(|x|+t)^2 \tag{A.20}$$

for all $t\geqslant 0$ and $(x,\xi)\in\Gamma^+(\tilde{R}_0,J_0,\sigma_0)$.

Now we introduce the set

$$I(x,\xi) = \{t \geqslant 0 \mid 2|\dot{y}(s)|^2 - C\langle y(s)\rangle^{-\rho} \geqslant 2(1-\delta_0)|v(\xi)|^2, \ \forall s \in [0,t]\}$$

and we shall prove that $T(x,\xi) := \sup I(x,\xi)$ is $+\infty$, for $(x,\xi) \in \Gamma^+(\tilde{R}_0, J_0, \sigma_0)$, possibly after increasing $\tilde{R}_0$. First, we remark that, for $t = 0$, $2|\dot{y}|^2 - C\langle y\rangle^{-\rho} \geqslant 2(1-\delta_0/2)|v(\xi)|^2$ by choosing $\tilde{R}_0$ large enough, which proves that $T(x,\xi) > 0$. Thus, for all $t \in [0, T(x,\xi))$ we get, integrating two times (A.19) on $[0,t]$ yields

$$|y(t,x,\xi)|^2 \geqslant e_0(|x| + t)^2 \tag{A.21}$$

using (A.20) and the definition of $I(x,\xi)$. This proves that $|\dot{y}(t)|^2 = v(\xi) + \mathcal{O}(\langle x\rangle^{-\rho})$, and then by increasing $\tilde{R}_0$ (in a way depending only on J_0, C and a finite number of semi-norms of w) we get for all $t \in [0, T(x,\xi))$

$$|\dot{y}(t)|^2 - C\langle y(t)\rangle^{-\rho} \geqslant 2(1-\delta_0/2)|v(\xi)|^2$$

which proves that $|\dot{y}|^2 - C\langle y\rangle^{-\rho} \geqslant (1-\delta_0)|v(\xi)|^2$ on $[0, T'(x,\xi))$ for some $T'(x,\xi) > T(x,\xi)$. This shows that $T(x,\xi) = +\infty$ and that (A.21) holds on $\mathbb{R}_+$. The result follows easily by integrating (A.18) on $[0,t]$, taking (A.21) into account. $\square$

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